Partial Differential Equations of Mixed Type
Lecture II

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Linear Partial Differential Equations

- **General Second-Order Equations of Mixed Type**

$$a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} = 0$$

Let $\lambda_1(x,y)$ and $\lambda_2(x,y)$ be two eigenvalues of $(a_{ij}(x,y))_{2\times 2}$

Mixed Hyperbolic-Elliptic Type: $\lambda_1(x,y)\lambda_2(x,y)$ **changes sign**

- **Fundamental Equations of Mixed Type**

Lavrentyev-Bitsadze Equation: $u_{xx} + \mathrm{sign}(x)u_{yy} = 0$

Tricomi Equation: $u_{xx} + xu_{yy} = 0$ (hyperbolic degeneracy at $x = 0$)

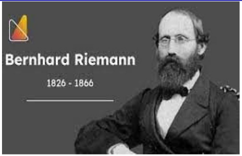
Keldysh Equation: $xu_{xx} + u_{yy} = 0$ (parabolic degeneracy at $x = 0$)

* **Euler-Poisson-Darboux Equation, Beltrami Equation,** $\cdots$

* **Fuchs-type PDEs,** $\cdots$

Riemann Problem: Bernhard Riemann 1860

Über die Fortpflanzung ebener Luftvellen von endlicher Schwingungsweite.
Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen, 8 (1860), 43–65.



Isentropic Euler Equations in one space dimension:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\big(\rho u^2 + p(\rho)\big) = 0, \end{cases}$$

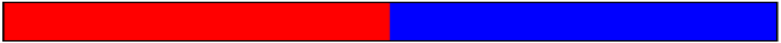
with the constitutive **pressure-density relation** $p(\rho)$, *e.g.*, after scaling:

$$p(\rho) = \rho^\gamma/\gamma \qquad \text{for adiabatic exponent } \gamma > 1.$$

Riemann Problem – The Riemann initial data take the form:

$$(\rho, u)(0, x) = \begin{cases} (\rho_L, u_L) & \text{for } x < 0, \\ (\rho_R, u_R) & \text{for } x > 0. \end{cases}$$

- The simplest initial value problem with discontinuous initial data that are piecewise constant and invariant under the self-similar scaling.



- Solutions of the 1-D Riemann problem consist of combinations of two-type elementary waves: shocks and centred rarefaction waves.

Hyperbolic Conservation Laws:

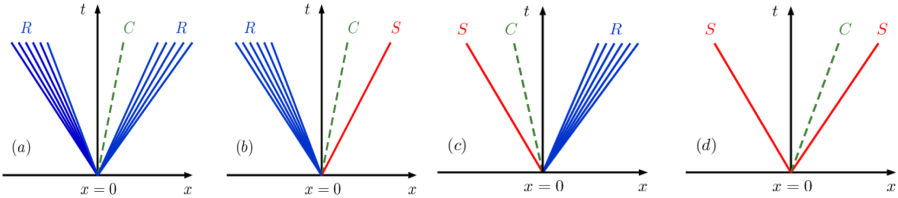
$$\partial_t U + \partial_x \mathbf{F}(U) = 0 \qquad \text{for } x \in \mathbb{R}.$$

Riemann Problem:

$$U(0,x) = \begin{cases} U_L & \text{for } x < 0, \\ U_R & \text{for } x > 0. \end{cases}$$

- **The general one-dimensional Riemann problem** was first solved by **Lax in 1957** with desired estimates under reasonable structural hypotheses.

  $\Longrightarrow$ Various extensions $\cdots\cdots$
- **Riemann solutions** consist of combinations of three-type elementary waves: **shocks, centred rarefaction waves, contact discontinuities.**



*Peter D. Lax: Hyperbolic Systems of Conservation Laws. II. Communications on Pure & Applied Mathematics, **10** (1957), 537–566.

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  $\Longrightarrow$ **Building blocks of the Glimm scheme (1965), the Lax-Friedrichs scheme (1954), the Godunov scheme (1959), wave front-tracking schemes,** $\cdots\cdots$.

  $\Longrightarrow$ **Existence theory of entropy solutions** – weak solutions satisfying the entropy conditions – for the general initial value problem in BV or L^∞.

- **Riemann solutions determine the local structure, the asymptotic states, and the global attractors of general entropy solutions**.

See Books: Dafermos 2016, Chen-Feldman 2018, Liu 2021, $\cdots$.

Bow Shock in Space generated by a Solar Explosion

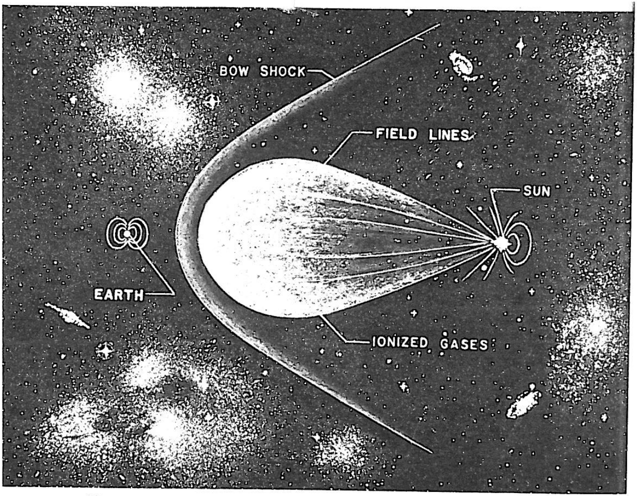
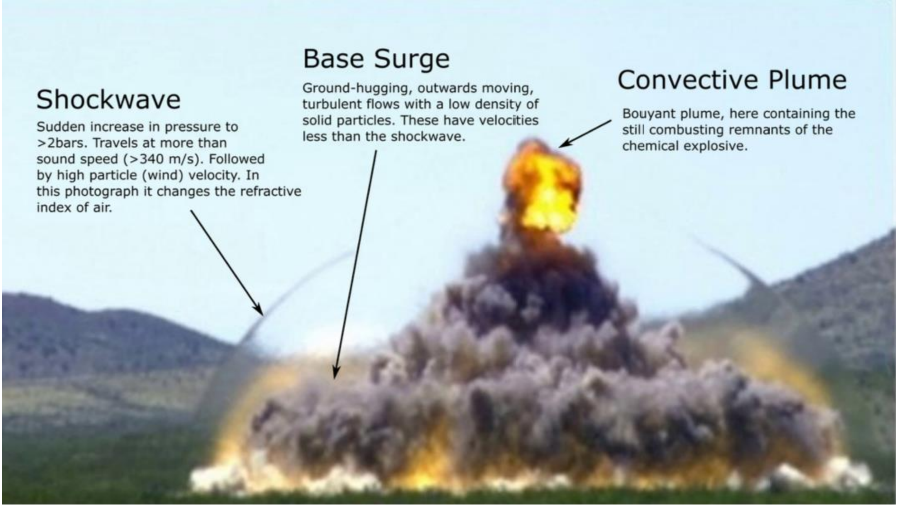


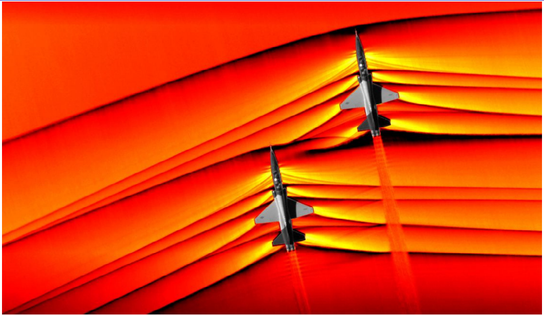
FIG. 50: SOLAR EXPLOSION

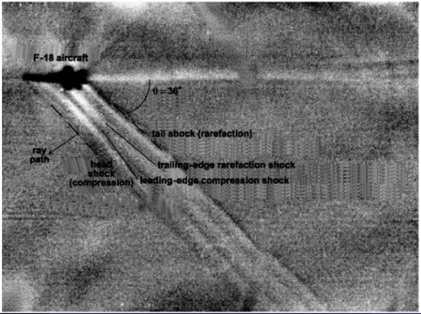
A shock wave in space generated by a solar eruption. The sketch shows the fully ionized nucleons attached to the solar magnetic field lines acting as the driving piston for the shock wave. (Courtesy: UTIAS, after Gold, 1962).

Blast Wave from a TNT Surface Explosion



Shock Waves generated by Aircrafts

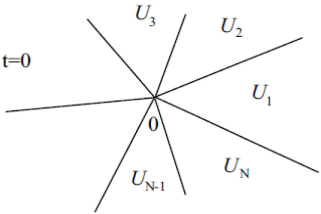




2-D Riemann Problem for Hyperbolic Conservation Laws

$$\partial_t U + \nabla_{\mathbf{x}} \cdot \mathbf{F}(U) = 0, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$

$$\partial_t \mathbf{A}(\Phi_t, \nabla_{\mathbf{x}}\Phi, \Phi) + \nabla_{\mathbf{x}} \cdot \mathbf{B}(\Phi_t, \nabla_{\mathbf{x}}\Phi, \Phi) = 0, \quad \nabla_{\mathbf{x}}\Phi = U$$



Books and Survey Articles: Asymptotic States and Global Attractors,...
Chang-Hsiao 1989, Glimm-Majda 1991, Li-Zhang-Yang 1998, Zheng 2001
Serre 2005, Chen 2005, Dafermos 2016, Chen-Feldman 2018, Chen 2023, $\cdots$

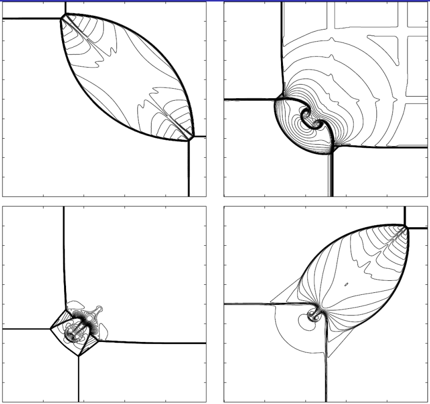
Numerical Solutions: Glimm-Klingenberg-McBryan-Plohr-Sharp-Yaniv 1985
Schulz-Rinne-Collins-Glaz 1993, Chang-Chen-Yang 1995, 2000,
Lax-Liu 1998, Kurganov-Tadmor 2002, $\cdots$

Theoretical Roles: Asymptotic States and Attractors
Local Structure and Building Blocks, $\cdots$

Classification of 2-D Riemann Problems for the Euler Eqs.

(a) Four shocks (Case 4)

(b) Two shocks and two vortex sheets (Case 11)



(c) Four shocks (Case 3)

(d) Two shocks and two vortex sheets (Case 12)

Figure: Numerical solutions to four (of nineteen) distinct cases of the 2D Riemann problem. Figures reproduced from Lax–Liu 1998.

- **Classification**: Zhang-Zheng 1990, Chang-Chen-Yang 1995, 2000, Li-Zhang-Yang 1998, Lax-Liu 1998, $\cdots$.
- **Rigorous Analysis for Solvability**: **Wide Open!**
  $\Longleftarrow$ **Free Boundary Problems for Nonlinear PDEs of Mixed Type**!

Euler Equations for Potential Flow: $(u, v) = \nabla_{\mathbf{x}}\Phi$ – Velocity

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \nabla_{\mathbf{x}} \Phi) = 0 & \textbf{(Conservation of mass)} \\ \partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 + h(\rho) = B & \textbf{(Bernoulli's law)} \end{cases}$$

with $h(\rho) = \frac{\rho^{\gamma-1}}{\gamma-1}$ for the pressure exponent $\gamma > 1$ for $p(\rho) = \frac{\rho^\gamma}{\gamma}$.

or, equivalently,

$$\partial_t \rho(\partial_t \Phi, \nabla_{\mathbf{x}}\Phi) + \nabla_{\mathbf{x}} \cdot \big(\rho(\partial_t \Phi, \nabla_{\mathbf{x}}\Phi)\nabla_{\mathbf{x}}\Phi\big) = 0,$$

with $\rho(\partial_t \Phi, \nabla_{\mathbf{x}}\Phi) = h^{-1}(B - \partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2).$

- Aerodynamics/Gas Dynamics: Fundamental PDE
- The potential flow equations and the full Euler equations coincide in important regions of the solution in this problem.
- J. Hadamard: Leçons sur la Propagation des Ondes, Hermann: Paris 1903
  P.-L. Lions: Mathematical Topics in Fluid Mechanics, Oxford 1996, 1998

  Majda-Thomann: CPDE 1987, Morawetz: CPAM 1994, $\cdots$

Riemann Problem with Four-Shock Interactions:

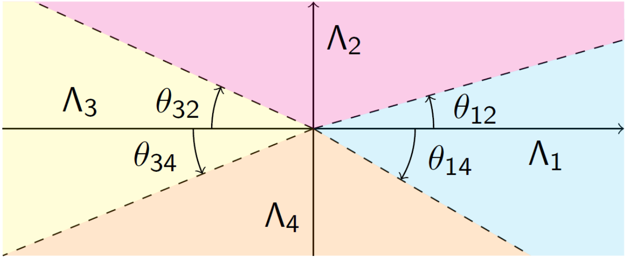
Riemann Initial Condition:

$$(\rho, \nabla_{\mathbf{x}}\Phi)|_{t=0} = (\rho_i, u_i, v_i), \qquad \mathbf{x} = (x_1, x_2) \in \Lambda_i,\ i = 1, 2, 3, 4.$$

- Initial data chosen to generate exactly four planar shocks
  $\longrightarrow$ State (2) fixed, other states become functions of angles
- $\max\{\rho_1, \rho_3\} < \min\{\rho_2, \rho_4\}$

Invariant under the Self-Similar Scaling:

$$(t, \mathbf{x}) \longrightarrow (\alpha t, \alpha \mathbf{x}), \quad (\rho, \Phi) \longrightarrow (\rho(\alpha t, \alpha \mathbf{x}), \tfrac{\Phi(\alpha t, \alpha \mathbf{x})}{\alpha}) \qquad \text{for } \alpha \neq 0$$



Seek Self-Similar Solutions: $(\xi,\eta)=(\frac{x_1}{t},\frac{x_2}{t}),\ \ D=(\partial_\xi,\partial_\eta)$

$$\rho(t,\mathbf{x})=\rho(\xi,\eta),\ \ \Phi(t,\mathbf{x})=t\Big(\varphi(\xi,\eta)+\frac{1}{2}(\xi^2+\eta^2)\Big)$$

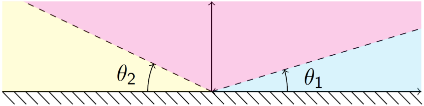
$$\mathrm{div}\,(\rho(D\varphi,\varphi)D\varphi)+2\rho(D\varphi,\varphi)=0$$

- **Elliptic:** $|D\varphi|<c_*(\varphi,B):=\sqrt{\frac{2(\gamma-1)}{\gamma+1}(B-\varphi)}$
- **Hyperbolic:** $|D\varphi|>c_*(\varphi,B):=\sqrt{\frac{2}{\gamma+1}(B-\varphi)}$

Second-Order Nonlinear Equations of Mixed Elliptic-Hyperbolic Type

Riemann Problem with Four-Shock Interactions
Symmetric Case: $\theta_{12} = \theta_{14} =: \theta_1$, $\theta_{32} = \theta_{34} =: \theta_2$, $\rho_2 = \theta_4$

In this case, the horizontal axis becomes a rigid wall $\Gamma_{\rm sym} = \{\eta = 0\}$.



Boundary Value Problem in the Coordinates (ξ, η):

Slip Boundary Condition on $\Gamma_{\rm sym}$: $D\varphi \cdot \boldsymbol{\nu} = 0$ on $\Gamma_{\rm sym}$.

Asymptotic Boundary Condition as $r := \sqrt{\xi^2 + \eta^2} \to \infty$:

$$D\varphi - (u_1 - \xi, v_1 - \eta) \to 0 \qquad 0 < \eta < \xi \tan\theta_1, \xi > 0,$$
$$D\varphi - (u_2 - \xi, v_2 - \eta) \to 0 \qquad -\eta \cot\theta_2 < \xi < \eta \cot\theta_1, \eta > 0,$$
$$D\varphi - (u_3 - \xi, v_3 - \eta) \to 0 \qquad 0 < \eta < \xi \tan\theta_2, \xi < 0,$$

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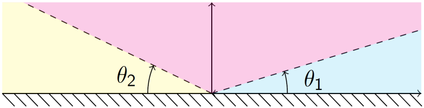

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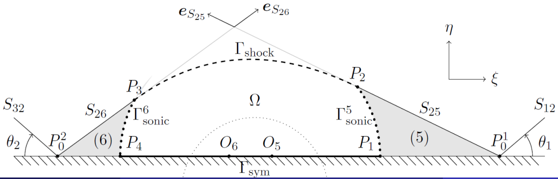
$$D\varphi - (u_3 - \xi, v_3 - \eta) \to 0 \qquad 0 < \eta < \xi \tan\theta_2, \xi < 0,$$

Locations of Shocks S_{26} **and** S_{25} **are** ***apriori*** **known when** $\theta_i \in (0, \theta_{\rm sonic})$, $i = 1, 2$.



Shocks: Rankine-Hugoniot (R-H) Conditions

Shocks are discontinuities in the pseudo-velocity $D\varphi$: If

- Ω^+ and $\Omega^- := \Omega \setminus \Omega^+$ are nonempty and open.
- $S := \partial\Omega \cap \Omega$ is a C^1–curve where $\nabla\varphi$ has a jump,

  then $\varphi \in C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution in Ω.

$\Longleftrightarrow$ φ satisfies

- The potential flow equation in $\Omega^\pm$.
- The Rankine-Hugoniot (R-H) conditions on S:

$$[\varphi]_S = 0,$$
$$[\rho(D\varphi, \varphi)D\varphi \cdot \boldsymbol{\nu}]_S = 0,$$

  where $[\,\cdot\,]_S$ is the jump of the quantity across S.

Entropy Condition: *The density function $\rho(D\varphi, \varphi)$ increases across a shock in the pseudo-flow direction*.

The entropy condition indicates that **the normal derivative function** $D\varphi \cdot \boldsymbol{\nu}$ on a shock always **decreases across the shock in the pseudo-flow direction**.

Constant Density State and Sonic Arcs

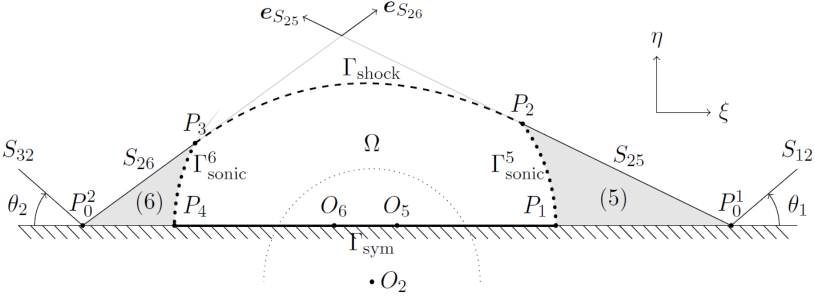
Constant Density State: If $\rho = \rho_0$ (a constant), then

$$\varphi(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) + u_0\xi + v_0\eta + k_0 \qquad \text{for } u_0, v_0, k_0 \in \mathbb{R}$$

Sonic Arcs: If $\rho = \rho_0$, then the flow is sonic, that is, $M = 1$ iff

$$|D\varphi| = c_0 := \rho_0^{\frac{\gamma-1}{2}} \quad \Longleftrightarrow \quad |(\xi, \eta) - (u_0, v_0)| = c_0$$

The circle $\partial B_{c_0}(u_0, v_0)$ is called the sonic circle of the constant density state $\rho = \rho_0$.



Find a curved shock Γ_{shock} and a function φ defined in region Ω, enclosed by $\Gamma^5_{\text{sonic}}, \Gamma_{\text{shock}}, \Gamma^6_{\text{sonic}}$, and $\Gamma_{\text{sym}} := \{\eta = 0\}$ such that φ satisfies

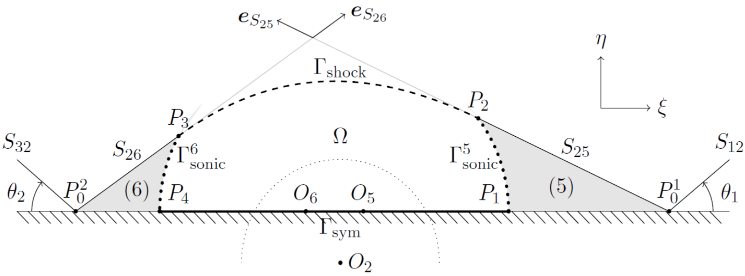
(i) **Equation (*)** and **Subsonicity** in Ω;

(ii) **Free Boundary Conditions**: $\varphi = \varphi_2$, $\rho D\varphi \cdot \boldsymbol{\nu}_{\text{s}} = D\varphi_2 \cdot \boldsymbol{\nu}_{\text{s}}$ on Γ_{shock};

(iii) $\varphi = \varphi_i$, $D\varphi = D\varphi_i$ on $\Gamma^i_{\text{sonic}}, i = 5, 6$;

(iv) $D\varphi \cdot \boldsymbol{\nu}_{\text{sym}} = 0$ on Γ_{sym},

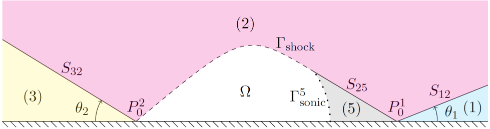
where $\boldsymbol{\nu}_{\text{s}}$ and $\boldsymbol{\nu}_{\text{sym}}$ are the interior unit normals on Γ_{shock} and Γ_{sym} resp.



***Caffarelli, Alt-Caffarelli-Friedman, Kinderlehrer-Nirenberg, Caffarelli-Jerison-Kenig, Figalli ⋯**

Mathematical Challenges

- **Nonlinear PDEs of Mixed Elliptic-Hyperbolic Type**
  - The transition boundary between the elliptic and hyperbolic phases is a priori unknown, so that most of the classical approaches, especially the fundamental solution approach, no longer work
- **New Approaches for Free Boundary Problems**
- **Optimal Estimates of Solutions to Nonlinear Degenerate PDEs**
  - Nonlinear elliptic degenerate PDEs (Keldysh-type degeneracy, ...)
  - Match of two boundary conditions
- **Corner Singularities** (Nonlinear PDEs without growth conditions)
  - Corner formed by the reflected-diffracted shock (free boundary) and the sonic arc (degenerate elliptic curve)
  - Corner between the reflected shock and the wedge at the reflection point for the transition from the supersonic to subsonic reflected-diffraction configuration when the wedge angle decreases.
- **Geometric Properties of Free Boundaries (Transonic Shocks)**



Theorem (Existence and Optimal Regularity of Weak Shock Solutions for All Incident Angles up to the Sonic Angle: Chen-Cliffe-Huang-Liu-Wang: JEMS 2024)

There is a unique sonic angle $\theta_{\rm sonic}$ *depending only on* (γ, v_2) *such that, when* $\theta_1, \theta_2 \in (0, \theta_{\rm sonic})$, *there exists a weak solution* φ *with* $\Gamma_{\rm shock}$ *such that*

(i) $\Gamma_{\rm shock} \subset \mathbb{R}^2_+ \setminus \overline{B_{c_2}(O_2)}$ *and* $\overline{S_{26} \cup \Gamma_{\rm shock} \cup S_{25}}$ *is* $C^{2,\alpha}$;

(ii) $\varphi \in C^\infty(\overline{\Omega} \setminus (\Gamma^5_{\rm sonic} \cup \Gamma^6_{\rm sonic})) \cap C^{2,\alpha}(\overline{\Omega} \setminus \{P_2, P_3\}) \cap C^{1,1}(\overline{\Omega})$;

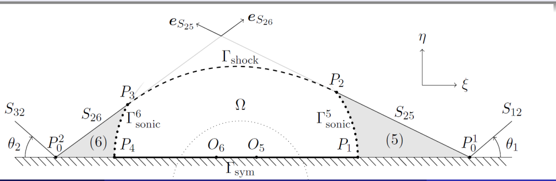
(iii) $|D\varphi| < c(|D\varphi|^2, \varphi)$ *in* Ω (*i.e., elliptic in* Ω);

(iv) $\max\{\varphi_5, \varphi_6\} \le \varphi \le \varphi_2$ *in* Ω;

(v) $\lim_{P \in \Omega, P \to P_*} (D_{rr}\varphi - D_{rr}\max\{\varphi_5, \varphi_6\}) = \frac{1}{\gamma + 1}$ *for any* $P_* \in (\Gamma^5_{\rm sonic} \cup \Gamma^6_{\rm sonic}) \setminus \{P_2, P_3\}$;

(vi) $\lim_{p \in \Omega, P \to \{P_1, P_2\}} D^2\varphi$ *do not exist.*

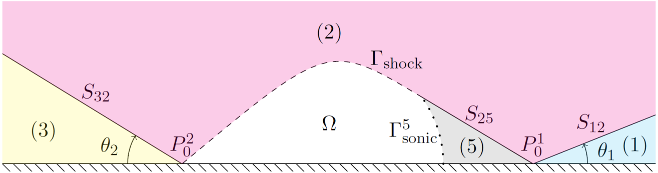
(v) $\varphi_\infty - \varphi$ *satisfies several important monotonicity properties*.*



Theorem (Beyond the Sonic Angle: θ_1 and/or $\theta_2 \in [\theta_{\rm sonic}, \theta_{\rm detach}]$ Chen-Cliffe-Huang-Liu-Wang: JEMS 2024; arXiv:2305.15224)

The Existence and Optimal Regularity Theorem still holds correspondingly, even when the incident angles θ_1 and/or θ_2 are between the sonic angle $\theta_{\rm sonic}$ and the detachment angle $\theta_{\rm detach} > \theta_{\rm sonic}$:

$$\theta_1 \in [\theta_{\rm sonic}, \theta_{\rm detach}) \quad \text{and/or} \quad \theta_2 \in [\theta_{\rm sonic}, \theta_{\rm detach})$$

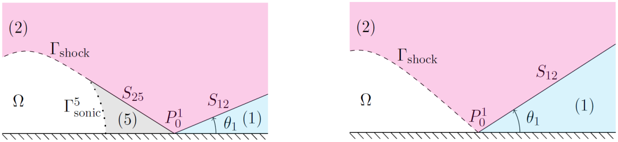


*The approach and related techniques have been developed based on the ideas/techniques from our earlier related work

Chen-Feldman 2018 (Research Monograph): The Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures, 832 pages, Annals of Mathematics Studies, 197, Princeton University Press, 2018

- **Strict monotonicity properties**
- **Uniform $C^{0,1}$ estimates of $\Omega, \Gamma_{\text{shock}}$ and φ** w.r.t. θ_1 and θ_2
- **Uniform estimate of ellipticity of the equation for φ**
- **Uniform weighted $C^{2,\alpha}$ estimates of φ (in Ω) and Γ_{shock}**
- **Define an iteration set $\mathcal{K}$ and an iteration mapping $\mathcal{I}$**
- **Show that $\mathcal{K} = [0, \theta_*] \times \mathcal{K}(\theta_{\rm w})$ and $\mathcal{F}$ satisfy the following:**
  (a) $\mathcal{F}$: $\mathcal{K} \subset [0, \theta_*] \times C^{2,\alpha}_* \to C^{2,\alpha}_*$ **is continuous**
  (b) $\mathcal{K}$ **is relatively open in** $[0, \theta_*] \times C^{2,\alpha}_*$
  (c) $\mathcal{F}(\theta_{\rm w}, \cdot) : \mathcal{K}(\theta_{\rm w}) \to \mathcal{K}(\theta_{\rm w})$ **has no fixed point on** $\partial\mathcal{K}(\theta_{\rm w})$
- **Show that** $\deg(\mathcal{F}(0, \cdot) - Id, \mathcal{K}(0), 0) \neq 0$

$\Longrightarrow \exists$ **a Fixed Point** φ (via the Leray-Schauder degree theory)

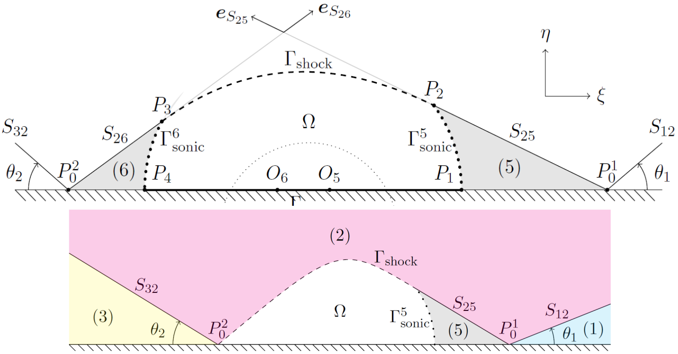


(a) Supersonic reflection, $\theta_1 \in (0, \theta^{\rm s})$ (b) Subsonic reflection, $\theta_1 \in [\theta^{\rm s}, \theta^{\rm d})$

Convexity of Transonic Shocks and Uniqueness/Stability

Chen-Feldman-Xiang (ARMA 2020): All of the transonic shocks in these problems are **uniformly convex** except Some Ending Points.

$\Longrightarrow$ Uniqueness and stability of global Riemann solutions with respect to the angles θ_1 and θ_2 (Preprint 2024)



*Caffarelli-Jerison-Kenig, Caffarelli-Salazar, Caffarelli-Spruck, Dolbeault-Monneau, Evans-Spruck, Plotnikov-Toland, $\cdots\cdots$

Self-Similar Solutions for the Full Euler Equations

$(u, v, p, \rho)(t, \mathbf{x}) = (U, V, p, \rho)(\xi_1, \xi_2), \;\; (\xi_1, \xi_2) = \frac{\mathbf{x}}{t}$

$$\begin{cases} \partial_{\xi_1}(\rho U) + \partial_{\xi_2}(\rho V)_{\xi_2} + 2\rho = 0, \\ \partial_{\xi_1}(\rho U^2 + p) + \partial_{\xi_2}(\rho UV) + 3\rho U = 0, \\ \partial_{\xi_1}(\rho UV) + \partial_{\xi_2}(\rho V^2 + p) + 3\rho V = 0, \\ \partial_{\xi_1}\big(U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1})\big) + \partial_{\xi_2}\big(V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1})\big) + 2\big(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}\big) = 0, \end{cases}$$

where $q = \sqrt{U^2 + V^2}$ and $(U, V) = (u, v) - \boldsymbol{\xi}$ is the pseudo-velocity.

Choose $W = (U, V, p, \rho)$ as the state variable. Then the system can be written as

$$\partial_{\xi_1} F(W) + \partial_{\xi_2} G(W) = H(W).$$

The eigenvalues, determined by $|\lambda \nabla_W F(W) - \nabla_W G(W)| = 0$, are

$$\lambda_0 = \frac{V}{U}(\text{repeated}), \qquad \lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2},$$

where $c = \sqrt{\gamma p / \rho}$ is the **sonic speed**.

Self-Similar Solutions for the Full Euler Equations

$(u, v, p, \rho)(t, \mathbf{x}) = (U, V, p, \rho)(\xi_1, \xi_2), \ \ (\xi_1, \xi_2) = \frac{\mathbf{x}}{t}$

$$\begin{cases} (\rho U)_{\xi_1} + (\rho V)_{\xi_2} + 2\rho = 0, \\ (\rho U^2 + p)_{\xi_1} + (\rho UV)_{\xi_2} + 3\rho U = 0, \\ (\rho UV)_{\xi_1} + (\rho V^2 + p)_{\xi_2} + 3\rho V = 0, \\ \big(U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1})\big)_{\xi_1} + \big(V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1})\big)_{\xi_2} + 2\big(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}\big) = 0, \end{cases}$$

where $q = \sqrt{U^2 + V^2}$ and $(U, V) = (u, v) - \boldsymbol{\xi}$ is the pseudo-velocity.

Eigenvalues: $\lambda_0 = \frac{V}{U}$(repeated), $\lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2}$,

where $c = \sqrt{\gamma p / \rho}$ is the **sonic speed**

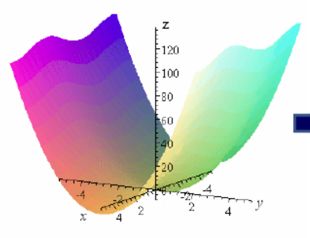
When the flow is pseudo-subsonic: $q < c$**,** the system consists of

- 2-transport equations: Compressible vortex sheets
- 2-nonlinear equations of mixed hyperbolic-elliptic type:

  Two kinds of transonic flow: Transonic shocks and sonic curves

*G.-Q. Chen: Two-Dimensional Riemann Problems:
Transonic Shock Waves and Free Boundary Problems.
Commun. Appl. Math. Comput. 5 (2023), no. 3, 1015–1052.

Isometric Embedding Problems



g_{ij} ⟷ **Metric**

(First Fundamental Form: $\mathrm{I} = \sum g_{ij} dx^i dx^j$)

h_{ij} ⟷ **Curvatures**

(Second Fundamental Form: $\mathrm{II} = \sum h_{ij} dx^i dx^j$)

Given a metric g_{ij} and certain curvatures

Inverse Problem: CAN we find a surface in our real world with this metric g_{ij} and corresponding curvatures?

Question: **Can even more sophisticated surfaces or thin sheets be realized in the Euclidean spaces?**

Fundamental:

- **Mathematics: Differential Geometry, Topology, ...**
- **Understanding evolution of sophisticated shapes of surfaces or thin sheets in nature, including**
  - **--Elasticity, Materials Sciences. ….**
  - **--Biology and Algorithmic Origami: Protein Folding, ….**
    - ***US DARPA's 10th question of the 23 Challenge Questions in the Sciences [US Defense Advanced Research Project Agency]:**
      - **Build a stronger mathematical theory for isometric and rigid embedding that can give insight into protein folding.**
- **Data Science, ….**
- **Human Design, Visual Arts, …..**

History: **Schlaefli (1873), Darboux (1894), Hilbert (1901), Weyl (1916), Janet (1926-27), Cartan (1926-27), Lewy (1936), Nash (1954-56), Kuiper (1955), Yau (1980's, 1990's), Gromov (1970, 1986), Günther (1989), Poznyak (1973), Levi (1908), Heinz (1962), Alexandroff (1938, 1942), Pogorelov (late 1940's, 1972), Nirenberg (1953, 1963), Efimov (1963), Bryant-Griffiths-Yan (1983), Lin (1985-86), Hong (1991,1993), Y. Li (1994), ……**

*Q. Han & J.-X. Hong: Isometric Embedding of Riemannian Manifolds in Euclidean Spaces, AMS, 2006

Nash Isometric Embedding Theorem (1956)
(C^k embedding theorem, $k \geq 3$)

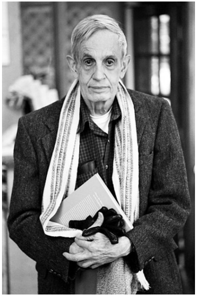
Every n-Dimensional Riemannian manifold (analytic or $C^k, k \geq 3$) can be C^k isometrically imbedded in the Euclidean space $\mathbb{R}^N$:

Compact Case: $N = 3s_n + 4n$

Noncompact Case: $N = (n+1)(3s_n + 4n)$

Gromov (1986): $N = s_n + 2n + 3$

Günther (1989): $N = \max\{s_n + 2n, s_n + n + 5\}$



Open Problems

Important for Applications

Rigidity of Isometric Embeddings?

Lowest Target Dimension? Janet-D: $N = s_n = \frac{n(n+1)}{2}$?

Optimal or Assigned Regularity: $C^{1,1}$, $BV(C^1)$, $W^{2,p}$, $\cdots$??

Efimov's Example (1966): No C^2–Isometric Embedding when $n = 2, s_n = 3$.

Isometric Embedding of Riemannian Manifolds $(\mathbf{M^2}, \mathrm{g})$ in $\mathbb{R}^3$

$\Omega \subset \mathbb{R}^2$ — Open set, $g = (g_{ij})$ — Given matric on $\mathbf{M^2}$.

Seek a map $\mathbf{r} : \Omega \to \mathbb{R}^3$ such that

$\mathrm{d}\mathbf{r} \cdot \mathrm{d}\mathbf{r} = g_{11}(\mathrm{d}x)^2 + 2g_{12}\mathrm{d}x\mathrm{d}y + g_{22}(\mathrm{d}y)^2 := I$ (1st fund. form)

$$\iff \quad \partial_x \mathbf{r} \cdot \partial_x \mathbf{r} = g_{11}, \quad \partial_x \mathbf{r} \cdot \partial_y \mathbf{r} = g_{12}, \quad \partial_y \mathbf{r} \cdot \partial_y \mathbf{r} = g_{22}$$

so that $(\partial_x \mathbf{r}, \partial_y \mathbf{r})$ in $\mathbb{R}^3$ are linearly independent.

The 2nd fundamental form:

$$II = -\mathrm{d}\mathbf{n} \cdot \mathrm{d}\mathbf{r} := h_{11}(\mathrm{d}x)^2 + 2h_{12}\mathrm{d}x\mathrm{d}y + h_{22}(\mathrm{d}y)^2$$

where $\mathbf{n} = \frac{\partial_x \mathbf{r} \times \partial_y \mathbf{r}}{|\partial_x \mathbf{r} \times \partial_y \mathbf{r}|}$ is the unit normal of the surface $\mathbf{r}(\Omega) \subset \mathbb{R}^3$.

Gauss-Codazzi System: Compatibility/Constraint

Fundamental Theorem in Differential Geometry: There exists a surface in $\mathbb{R}^3$ whose 1st and 2nd fundamental forms are I and II if the coefficients $\{g_{ij}\}$ and $\{h_{ij}\}$ of the two given quadratic forms I and II, I being positive definite, satisfy the Gauss-Codazzi system.

*This theorem holds even when $h_{ij} \in L^p$ (Ciarlet, Mardare, $\cdots$)

For given $\{g_{ij}\}$, $\{h_{ij}\}$ is determined by the Codazzi Equations (Compatibility):

$$\begin{cases} \partial_x M - \partial_y L = \Gamma_{22}^{(2)} L - 2\Gamma_{12}^{(2)} M + \Gamma_{11}^{(2)} N, \\ \partial_x N - \partial_y M = -\Gamma_{22}^{(1)} L + 2\Gamma_{12}^{(1)} M - \Gamma_{11}^{(1)} N, \end{cases}$$

and the Gauss Equation (Constraint):

$$LN - M^2 = K \qquad \text{(Monge-Ampère Constraint)}$$

where $\quad L = \frac{h_{11}}{\sqrt{|g|}},\ M = \frac{h_{12}}{\sqrt{|g|}},\ N = \frac{h_{22}}{\sqrt{|g|}},\ |g| = g_{11}g_{22} - g_{12}^2$

Christoffel symbols: $\Gamma_{ij}^{(k)} = \frac{1}{2} g^{kl}(\partial_j g_{il} + \partial_i g_{jl} - \partial_i g_{ij})$

Gauss curvature: $K(x,y) = \frac{R_{1212}}{|g|}$

Riemann curvatures: $R_{ijkl} = g_{lm}\big(\partial_k \Gamma_{ij}^{(m)} - \partial_l \Gamma_{ik}^{(m)} + \Gamma_{ij}^{(n)}\Gamma_{nk}^{(m)} - \Gamma_{ik}^{(n)}\Gamma_{nj}^{(m)}\big)$

Gauss-Codazzi System: Compatibility/Constraint

For given $\{g_{ij}\}$, $\{h_{ij}\}$ is determined by the Codazzi Equations (Compatibility):

$$\begin{cases} \partial_x M - \partial_y L = \Gamma_{22}^{(2)} L - 2\Gamma_{12}^{(2)} M + \Gamma_{11}^{(2)} N, \\ \partial_x N - \partial_y M = -\Gamma_{22}^{(1)} L + 2\Gamma_{12}^{(1)} M - \Gamma_{11}^{(1)} N, \end{cases}$$

and the Gauss Equation (Constraint):

$$LN - M^2 = K \qquad \text{(Monge-Ampère vonstraint)}$$

Consider $U := (M, N)^\top$ as the state variables. If $N \neq 0$, then the Gauss-Codazzi system can be written as

$$\partial_x U + \partial_y F(U) = L.O.T.$$

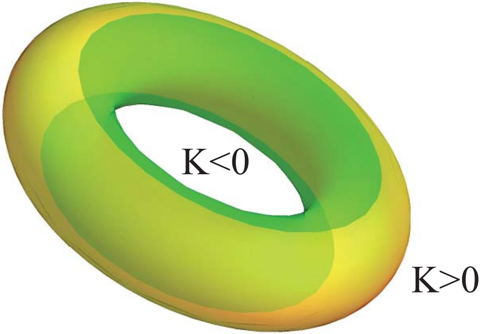
The eigenvalues of the system, determined by $|\lambda I - \nabla_U F(U)| = 0$, are

$$\lambda_\pm = \frac{-M \pm \sqrt{-K}}{N}.$$

Nonlinear PDEs of Mixed Elliptic-Hyperbolic Type: Sign of K

- Hyperbolic if $K < 0$.
- Elliptic if $K > 0$.
- Parabolic if $K = 0$.

Surfaces with Gauss Curvature of Changing Sign



Gauss Curvature K on a Torus:
Toroidal Shell or Doughnut Surface

Fluid Dynamics Formalism for Isometric Embedding

Set $\quad L = \rho v^2 + p, \quad M = -\rho uv, \quad N = \rho u^2 + p, \quad q^2 = u^2 + v^2.$

Choose p as the Chaplygin-type gas: $\quad p = -1/\rho.$

The Codazzi Equations become the Momentum Equations:

$$\begin{cases} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) = -\Gamma_{22}^{(2)}(\rho v^2 + p) - 2\Gamma_{12}^{(2)}\rho uv - \Gamma_{11}^{(2)}(\rho u^2 + p), \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) = -\Gamma_{22}^{(1)}(\rho v^2 + p) - 2\Gamma_{12}^{(1)}\rho uv - \Gamma_{11}^{(1)}(\rho u^2 + p), \end{cases}$$

and the Gauss Equation becomes the Bernoulli Relation: $p = -\sqrt{q^2 + K}.$

Define the sound speed: $c^2 = p'(\rho)$. Then $c^2 = 1/\rho^2 = q^2 + K$.

$c^2 > q^2$ and the "flow" is subsonic when $K > 0$,

$c^2 < q^2$ and the "flow" is supersonic when $K < 0$,

$c^2 = q^2$ and the "flow" is sonic when $K = 0$.

?? **Weak Continuity and Existence for the Gauss-Codazzi Equations**

$\Leftarrow$ **Weak Convergence Methods: Compensated Compactness**

Chen-Slemrod-Wang: Commun. Math. Phys. 2010

*Cao-Han-Huang-Wang (2023): $K < 0$ [surfaces with finite total curvature]

*Christoforou-Slemrod (2016), S. Li (2020), $\cdots$

*Acharya-Chen-Li-Slemrod-Wang (2017): 2D and 3D & Evolution Problems

Gauss-Codazzi-Ricci System for Isometric Embedding of d-D Riemannian Manifolds into $\mathbb{R}^N$: $d \geq 3$

Gauss Equations: $h^a_{ji}h^a_{kl} - h^a_{ki}h^a_{jl} = R_{ijkl}$

Codazzi Equations:

$$\frac{\partial h^a_{lj}}{\partial x^k} - \frac{\partial h^a_{kj}}{\partial x^l} + \Gamma^m_{lj}h^a_{km} - \Gamma^m_{kj}h^a_{lm} + \kappa^a_{kb}h^b_{lj} - \kappa^a_{lb}h^b_{kj} = 0$$

Ricci Equations:

$$\frac{\partial \kappa^a_{lb}}{\partial x^k} - \frac{\partial \kappa^a_{kb}}{\partial x^l} - g^{mn}\left(h^a_{ml}h^b_{kn} - h^a_{mk}h^b_{ln}\right) + \kappa^a_{kc}\kappa^c_{lb} - \kappa^a_{lc}\kappa^c_{kb} = 0$$

where R_{ijkl} is the Riemann curvature tensor, $\kappa^a_{kb} = -\kappa^b_{ka}$ is the coefficients of the connection form on the normal bundle; the indices a, b, c run from 1 to N, and i, j, k, l, m, n run from 1 to $d \geq 3$.

***The Gauss-Codazzi-Ricci system has no type,**
neither purely hyperbolic nor purely elliptic
for general Riemann curvature tensor R_{ijkl} (S.-S. Chern & H. Levy)

*Bryant-Griffiths-Yang (**1983**): Duke Math. J., 102 pages.

*Chen-Clelland-Slemrod-Wang-Yang (AJM **2018**): Positive Symmetric Systems

Weak Continuity/Rigidity of the Gauss-Codazzi-Ricci System

Theorem (Chen-Slemrod-Wang: Proc. Amer. Math. Soc. 2010)

Let $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ be a sequence of solutions to the Gauss-Codazzi-Ricci system, which is uniformly bounded in $L^p, p > 2$.

$\Longrightarrow$

The weak limit vector field $(h_{ij}^{a}, \kappa_{lb}^{a})$ of the sequence $(h_{ij}^{a,\varepsilon}, \kappa_{lb}^{a,\varepsilon})$ in L^p is still a solution to the Gauss-Codazzi-Ricci system.

Observations: Div-Curl Structure of the GCR System

$$\mathrm{div}\,(\overbrace{0,\cdots,0,h^{a,\varepsilon}_{lj}}^{k},\underbrace{,0,\cdots,-h^{a,\varepsilon}_{kj}}_{l},0,\cdots,0)=R_1,\quad \mathrm{curl}\,(h^{a,\varepsilon}_{1i},h^{a,\varepsilon}_{2i},\cdots,h^{a,\varepsilon}_{di})=R_2,$$

$$\mathrm{div}\,(\overbrace{0,\cdots,0,\kappa^{a,\varepsilon}_{lb}}^{k},\underbrace{,0,\cdots,-\kappa^{a,\varepsilon}_{kb}}_{l},0,\cdots,0)=R_3,\quad \mathrm{curl}\,(\kappa^{a,\varepsilon}_{1b},\kappa^{a,\varepsilon}_{2b},\cdots,\kappa^{a,\varepsilon}_{db})=R_4,$$

$$\mathrm{div}\,(\overbrace{0,\cdots,0,h^{b,\varepsilon}_{li}}^{k},\underbrace{,0,\cdots,-h^{b,\varepsilon}_{ki}}_{l},0,\cdots,0)=R_5,\quad \mathrm{curl}\,(h^{b,\varepsilon}_{1i},h^{b,\varepsilon}_{2i},\cdots,h^{b,\varepsilon}_{di})=R_6,$$

$$\mathrm{div}\,(\overbrace{0,\cdots,0,\kappa^{b,\varepsilon}_{lc}}^{k},\underbrace{,0,\cdots,-\kappa^{b,\varepsilon}_{kc}}_{l},0,\cdots,0)=R_7,\quad \mathrm{curl}\,(\kappa^{b,\varepsilon}_{1c},\kappa^{b,\varepsilon}_{2c},\cdots,\kappa^{b,\varepsilon}_{dc})=R_8.$$

Lemma (Classical Div-Curl Lemma: Murat-Tartar)

Let $\Omega \subset \mathbb{R}^d, d \geq 2,$ *be open bounded. Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that, for* $\varepsilon > 0$, *two fields* $\mathbf{u}^\varepsilon \in L^p(\Omega; \mathbb{R}^d)$, $\mathbf{v}^\varepsilon \in L^q(\Omega; \mathbb{R}^d)$ *satisfy the following*:

1. $\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}$ *weakly in* $L^p(\Omega; \mathbb{R}^d)$ *as* $\varepsilon \to 0$;
2. $\mathbf{v}^\varepsilon \rightharpoonup \mathbf{v}$ *weakly in* $L^q(\Omega; \mathbb{R}^d)$ *as* $\varepsilon \to 0$;
3. $\operatorname{div} \mathbf{u}^\varepsilon$ *are confined in a compact subset of* $W^{-1,p}_{\mathrm{loc}}(\Omega; \mathbb{R})$;
4. $\operatorname{curl} \mathbf{v}^\varepsilon$ *are confined in a compact subset of* $W^{-1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^{d\times d})$.

Then the scalar product of $\mathbf{u}^\varepsilon$ *and* $\mathbf{v}^\varepsilon$ *are weakly continuous:*

$$\mathbf{u}^\varepsilon \cdot \mathbf{v}^\varepsilon \longrightarrow \mathbf{u} \cdot \mathbf{v} \qquad \text{in the sense of distributions.}$$

***Various variations of this lemma for different applications/purposes**:

Robbin-Rogers-Temple (1987)

Kozono-Yanagisawa (2009)

Cont-Dolzmann-Müller (2011)

...

$$\text{div}\,(\overbrace{0,\cdots,0,h^{a,\varepsilon}_{lj}}^{k},\underbrace{0,\cdots,-h^{a,\varepsilon}_{kj}}_{l},0,\cdots,0)=R_1,\quad \text{curl}\,(h^{a,\varepsilon}_{1i},h^{a,\varepsilon}_{2i},\cdots,h^{a,\varepsilon}_{di})=R_2,$$

$$\text{div}\,(\overbrace{0,\cdots,0,\kappa^{a,\varepsilon}_{lb}}^{k},\underbrace{0,\cdots,-\kappa^{a,\varepsilon}_{kb}}_{l},0,\cdots,0)=R_3,\quad \text{curl}\,(\kappa^{a,\varepsilon}_{1b},\kappa^{a,\varepsilon}_{2b},\cdots,\kappa^{a,\varepsilon}_{db})=R_4,$$

$$\text{div}\,(\overbrace{0,\cdots,0,h^{b,\varepsilon}_{li}}^{k},\underbrace{0,\cdots,-h^{b,\varepsilon}_{ki}}_{l},0,\cdots,0)=R_5,\quad \text{curl}\,(h^{b,\varepsilon}_{1i},h^{b,\varepsilon}_{2i},\cdots,h^{b,\varepsilon}_{di})=R_6,$$

$$\text{div}\,(\overbrace{0,\cdots,0,\kappa^{b,\varepsilon}_{lc}}^{k},\underbrace{0,\cdots,-\kappa^{b,\varepsilon}_{kc}}_{l},0,\cdots,0)=R_7,\quad \text{curl}\,(\kappa^{b,\varepsilon}_{1c},\kappa^{b,\varepsilon}_{2c},\cdots,\kappa^{b,\varepsilon}_{dc})=R_8.$$

Weak Convergence: Div-Curl $\Rightarrow$

$$\begin{aligned} h^{a,\varepsilon}_{lj}h^{b,\varepsilon}_{ki}-h^{a,\varepsilon}_{kj}h^{b,\varepsilon}_{li} &\rightharpoonup h^{a}_{lj}h^{b}_{ki}-h^{a}_{kj}h^{b}_{li},\\ \kappa^{a,\varepsilon}_{kb}\kappa^{b,\varepsilon}_{lc}-\kappa^{a,\varepsilon}_{lb}\kappa^{b,\varepsilon}_{kc} &\rightharpoonup \kappa^{a}_{kb}\kappa^{b}_{lc}-\kappa^{a}_{lb}\kappa^{b}_{kc}\\ \kappa^{a,\varepsilon}_{kb}h^{b,\varepsilon}_{li}-\kappa^{a,\varepsilon}_{lb}h^{b,\varepsilon}_{ki} &\rightharpoonup \kappa^{a}_{kb}h^{b}_{li}-\kappa^{a}_{lb}h^{b}_{ki}\end{aligned}$$

in the sense of distributions as $\varepsilon \to 0$

Compensated Compactness Theorem on Banach Spaces

$\mathcal{H}$ – Hilbert space over field $\mathcal{K}$ with $\mathcal{H} = \mathcal{H}^*$

Y, Z – Reflexive Banach space over $\mathbb{K}$ with dual spaces Y^*, Z^*

Theorem (Chen-S. Li: J. Geometric Analysis, 2018)

Let $S : \mathcal{H} \to Y$ *with adjoint operator* $S^\dagger : Y^* \to \mathcal{H}$,
$T : \mathcal{H} \to Z$ *with adjoint operator* $T^\dagger : Z^* \to \mathcal{H}$ *satisfy*

- *Orthogonality*: $S \circ T^\dagger = 0, \quad T \circ S^\dagger = 0$;
- *For some Hilbert space* $\underline{\mathcal{H}}$ *so that* $\mathcal{H}$ *embeds compactly into* $\underline{\mathcal{H}}$, *there exists* $C > 0$ *such that, for all* $\mathbf{h} \in \mathcal{H}$,
$$\|\mathbf{h}\|_{\mathcal{H}} \leq C\big(\|S\mathbf{h}\|_Y + \|T\mathbf{h}\|_Z + \|\mathbf{h}\|_{\underline{\mathcal{H}}}\big).$$

Assume that two sequences $\{\mathbf{u}^\varepsilon\}, \{\mathbf{v}^\varepsilon\} \Subset \mathcal{H}$ *satisfy*

- $\mathbf{u}^\varepsilon \rightharpoonup \bar{\mathbf{u}}$ *and* $\mathbf{v}^\varepsilon \rightharpoonup \bar{\mathbf{v}}$ *in* $\mathcal{H}$ *as* $\varepsilon \to 0$;
- $\{S\mathbf{u}^\varepsilon\}$ *is pre-compact in* Y, *and* $\{T\mathbf{v}^\varepsilon\}$ *is pre-compact in* Z.

Then, after passing to subsequences if necessary,
$$\langle \mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon\rangle_{\mathcal{H}} \longrightarrow \langle \bar{\mathbf{u}}, \bar{\mathbf{v}}\rangle_{\mathcal{H}} \qquad \text{as } \varepsilon \to 0.$$

$\Longrightarrow$ Chen-Li (2018): **Intrinsic Div-Curl Lemma on Riemannian Manifolds**

*Chen-Giron (2024): Non-abelian Div-Curl Lemma

Global Weak Rigidity of the Gauss-Codazzi-Ricci Equations

Theorem (Chen-S. Li: J. Geometric Analysis, 2018)

- **Let** (M, g) **be a Riemannian manifold with** $g \in W^{1,p}$ **for** $p > 2$**.**
- **Let** $(\boldsymbol{h}^\varepsilon, \boldsymbol{\kappa}^\varepsilon)$ **be a sequence of solutions** ($\sim$ *coefficients of the 2nd fundamental form and the connection form on the normal bundle*) **in** L^p **of the GCR equations in the distributional sense.**

Assume that, for any submanifold $K \Subset M$**, there exists** $C_K > 0$ **independent of** ε **such that**

$$\sup_{\epsilon>0} \left\{ \|\boldsymbol{h}^\varepsilon\|_{L^p(K)} + \|\boldsymbol{\kappa}^\varepsilon\|_{L^p(K)} \right\} \leq C_K.$$

$\Longrightarrow$ **When** $\varepsilon \to 0$**, there exists a subsequence of** $(\boldsymbol{h}^\varepsilon, \boldsymbol{\kappa}^\varepsilon)$ **that converges weakly in** L^p **to a pair** $(\boldsymbol{h}, \boldsymbol{\kappa})$ **that is still a weak solution of the GCR equations**.

$\Longrightarrow$ **Global Weak Rigidity of Isometric Immersions in** $W^{2,p}$

*Globally, independent of local coordinates

*No restriction on the Riemann curvatures and the types of PDEs

*The Cartan Formalism: Similar

Global Weak Continuity of the Gauss-Codazzi-Ricci Eqs.

Theorem (Chen-Giron 2024)

- **Let (M, g) be a Riemannian manifold with $g \in W^{1,p}$ for $p > 2$.**
- **Let $(\boldsymbol{h}^\varepsilon, \boldsymbol{\kappa}^\varepsilon)$ be a sequence of solutions** ($\sim$ *coefficients of the 2nd fundamental form and the connection form on the normal bundle*) **in L^p of the GCR equations in the distributional sense.**

Assume that, for any submanifold $K \Subset M$, there exists $C_K > 0$ independent of ε such that

$$\sup_{\epsilon > 0} \|\boldsymbol{h}^\varepsilon\|_{L^p(K)} \le C_K.$$

$\Longrightarrow$ **There exists a refined sequence $(\tilde{\boldsymbol{h}}^\varepsilon, \tilde{\boldsymbol{\kappa}}^\varepsilon)$ that are still weak solutions of the GCR equations such that, when $\varepsilon \to 0$, $(\tilde{\boldsymbol{h}}^\varepsilon, \tilde{\boldsymbol{\kappa}}^\varepsilon)$ converges weakly in L^p to a pair $(\boldsymbol{h}, \boldsymbol{\kappa})$ that is still a weak solution of the GCR equations**.

$\Longrightarrow$ **Global Weak Rigidity of Isometric Immersions in $W^{2,p}$**

*Globally, indept. of local coordinates;

*No restriction on the *Riemann curvatures* and the *types of PDEs*

**Invariance* for a choice of suitable gauge to control the *Full Connection Form*

Global Weak Rigidity on Manifolds with Lower Regularity: Global Analysis

- **Global Weak Continuity of the Gauss-Codazzi-Ricci Equations on Manifolds with Lower Regularity**:
  Chen-Li (JGA 2018), Chen-Giron (2024): Riemannian Manifolds
  Chen-Li (ARMA 2021): Semi-Riemannian Mflds (*e.g.* Lorentzian Mflds)
  ***A unified intrinsic geometric approach** indept. of the local coordinates.
- **Limiting Surfaces in Geometry**: **The weak limit of isometrically immersed surfaces is still an isometrically immersed surface in $\mathbb{R}^d$ governed by the GCR Eqs. for any R_{ijkl} without sign/type restriction**
  Chen-Li (JGA2018, ARMA2021), Chen-Li-Slemrod (JMPA2022), Chen-Giron (2024)
- **Motivations and Connections**:
  Theory of Polyconvexity in Nonlinear Elasticity: Ball, $\cdots$
  Intrinsic Methods in Elasticity & Nonlinear Korn Inequalities: Ciarlet, $\cdots$
  Concentration-Compactness Principles: Lions, $\cdots$
  Convex Integration & Flexibility: Gromov, De Lellis, Székelyhidi, $\cdots$
  Uhlenbeck Compactness, Immersions, $\cdots$: Chen-Giron 2024, $\cdots$
  *Uhlenbeck 1982, Donaldson 1983, $\cdots$, Reintjes-Temple 2020, $\cdots$

  Book: **Differential Geometry and Continuum Mechanics**
  By G.-Q. Chen, M. Grinfeld & R. Knops, Springer-Verlag, 2015

Concluding Remarks

Nonlinear Partial Differential Equations
of Mixed Type, or even No Type,

naturally arise in many fundamental problems in

Fluid Mechanics, Differential Geometry
Elasticity, Materials Science, Relativity
Optimization, Dynamical Systems, ...

The solution to these fundamental problems in the areas greatly requires a **deep understanding of**

Nonlinear Partial Differential Equations of
Mixed Type, esp. Elliptic-Hyperbolic Type
& Further Developments of New Mathematics.

***G.-Q. Chen: Partial Differential Equations of Mixed Type**
— Analysis and Applications
Notices of the American Mathematical Society, 70 (2023), no. 1, 8–23.