Partial Differential Equations of Mixed Type

Lecture I

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Three of the Fundamental Types: Representatives

Elliptic Equation: Laplace's Equation & Equilibrium Equation

$$\sum_{j=1}^{n} u_{x_j x_j} = 0$$

Parabolic Equation: Heat Equation

$$u_t - \sum_{j=1}^{n} u_{x_j x_j} = 0$$

Hyperbolic Equation: Wave Equation & Maxwell's Equation

$$u_{tt} - \sum_{j=1}^{n} u_{x_j x_j} = 0$$

Classification for Second-Order PDEs

Jacques Hadamard: **Lectures on Cauchy's Problem in Linear Partial Differential Equations**

Yale University Press, Oxford University Press, 1923

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Distinctions: Properties of Solutions

- **Infinite $\Longleftrightarrow$ Finite Speed of Propagation of Solutions**
- **Gain $\Longleftrightarrow$ Loss of Regularity of Solutions**
- $\cdots\cdots$

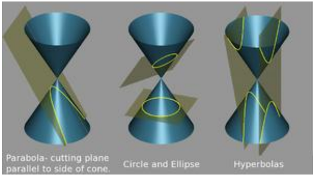
Classification for 2-D Const. Coeff. 2^{nd} Order PDEs

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = f$$

Let $\lambda_1 \le \lambda_2$ be two eigenvalues of the 2×2 symmetric matrix $(a_{ij})_{2\times 2}$.

Elliptic: $(a_{ij})_{2\times 2} > 0 \iff \lambda_1\lambda_2 > 0 \iff a_{12}^2 - a_{11}a_{22} < 0$

Hyperbolic: $(a_{ij})_{2\times 2} < 0 \iff \lambda_1\lambda_2 < 0 \iff a_{12}^2 - a_{11}a_{22} > 0$



- **Classification of Conic Sections and Quadratic Forms**:

$$a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2$$

Quadratic Curves: (i) Parabolas, (ii) Ellipses, (iii) Hyperbolas

- **Fourier Transform**: $(a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2)\,\hat{u} = -\hat{f}$

- **General Second-Order Equations of Mixed Type**

$$a_{11}(x,y)u_{xx} + 2a_{12}(x,y)u_{xy} + a_{22}(x,y)u_{yy} = 0$$

Let $\lambda_1(x,y)$ and $\lambda_2(x,y)$ be two eigenvalues of $(a_{ij}(x,y))_{2\times 2}$

Mixed Hyperbolic-Elliptic Type: $\lambda_1(x,y)\lambda_2(x,y)$ **changes sign**

- **Fundamental Equations of Mixed Type**

Lavrentyev-Bitsadze Equation: $u_{xx} + \text{sign}(x)u_{yy} = 0$

Tricomi Equation: $u_{xx} + xu_{yy} = 0$ (hyperbolic degeneracy at $x = 0$)

Keldysh Equation: $xu_{xx} + u_{yy} = 0$ (parabolic degeneracy at $x = 0$)

* **Euler-Poisson-Darboux Equation, Beltrami Equation,** $\cdots$

* **Fuchs-type PDEs,** $\cdots$

Linear Fundamental Equations of Mixed Type I

Lavrentyev-Bitsadze Equation:

$$\partial_{xx}u + \mathrm{sign}(x)\partial_{yy}u = 0$$

- When $x > 0 \Longrightarrow$ Laplace equation

$$\partial_{xx}u + \partial_{yy}u = 0$$

- When $x < 0 \Longrightarrow$ Wave equation

$$\partial_{xx}u - \partial_{yy}u = 0$$

- Transition boundary $x = 0$ between the Laplace equation and the wave equation:

Jump discontinuous coefficient $\mathrm{sign}(x)$.

Linear Fundamental Equations of Mixed Type II

Tricomi Equation:

$$u_{xx} + x u_{yy} = 0.$$

- When $x > 0 \Longrightarrow$ Elliptic equation

$$u_{xx} + x u_{yy} = 0, \qquad x > 0.$$

- When $x < 0 \Longrightarrow$ Hyperbolic equation

$$u_{xx} - |x| u_{yy} = 0, \qquad x < 0.$$

- **Hyperbolic degeneracy at** $x = 0$: The two characteristic families coincide perpendicularly to line $x = 0$. Its degeneracy is determined by the **Elliptic or Hyperbolic Euler-Poisson-Darboux Equation**:

$$u_{\tau\tau} \pm u_{yy} + \frac{\beta}{\tau} u_\tau = 0 \qquad \text{for } \pm x > 0 \qquad (\beta = \frac{1}{3},\, \tau = \frac{2}{3}|x|^{\frac{3}{2}}).$$

Linear Fundamental Equations of Mixed Type III

Keldysh Equation:

$$xu_{xx} + u_{yy} = 0.$$

- When $x > 0 \Longrightarrow$ Elliptic equation

$$xu_{xx} + u_{yy} = 0, \qquad x > 0.$$

- When $x < 0 \Longrightarrow$ Hyperbolic equation

$$|x|u_{xx} - u_{yy} = 0, \qquad x < 0.$$

- **Parabolic degeneracy at** $x = 0$: The two characteristic families are quadratic parabolas lying in half-plane $x < 0$ and tangential at contact points to the degenerate line $x = 0$:

$$u_{\tau\tau} \pm u_{yy} + \frac{\beta}{\tau} u_\tau = 0 \qquad \text{for } \pm x > 0 \qquad (\beta = -\frac{1}{4}, \, \tau = \frac{1}{2}|x|^{\frac{1}{2}}).$$

Nonlinear PDEs of Mixed Type: Simplest Model

$$u_{xx} + uu_{yy} = 0.$$

- **When** $u > 0 \Longrightarrow$ **Elliptic equation**
- **When** $u < 0 \Longrightarrow$ **Hyperbolic equation**
- **Transition boundary between the elliptic and hyperbolic phases:**
  $u(x, y) = 0$. **It is a free boundary in general!**

This is a nonlinear version of the linear PDEs of mixed type:

Tricomi Equation: $u_{xx} + xu_{yy} = 0$ ($u(x, y) = x$ near $x = 0$)

Keldysh Equation: $xu_{xx} + u_{yy} = 0$ ($u(x, y) = \frac{1}{x}$ near $x = 0$)

*Relation: The Transonic Small Disturbance Equation in fluid dynamics:

$$u_{xx} + (uu_y)_y = 0.$$

J. Hunter, C. Morawetz, B. Keyfitz, S. Canic, G. Lieberman, $\cdots$.

Airfoil Problems: Transonic flow past airfoils

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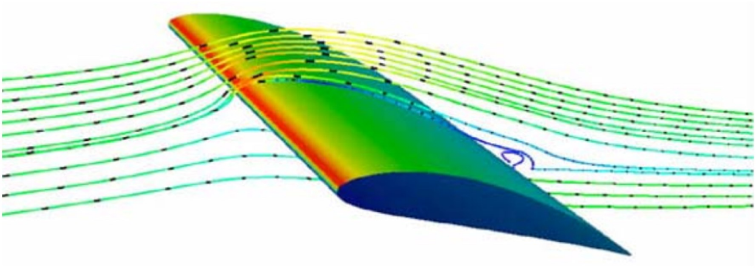
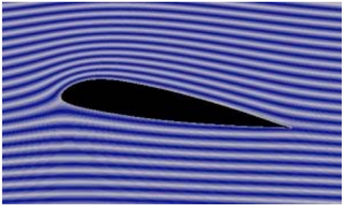
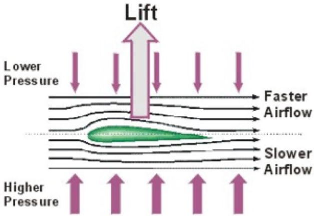
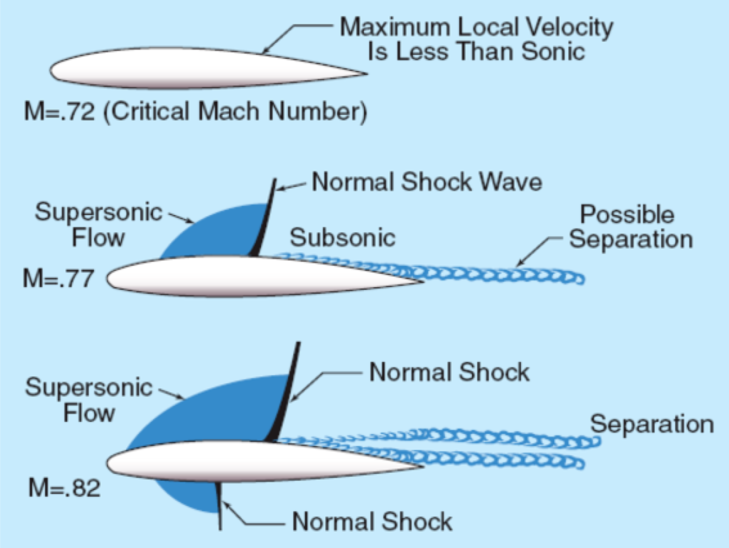


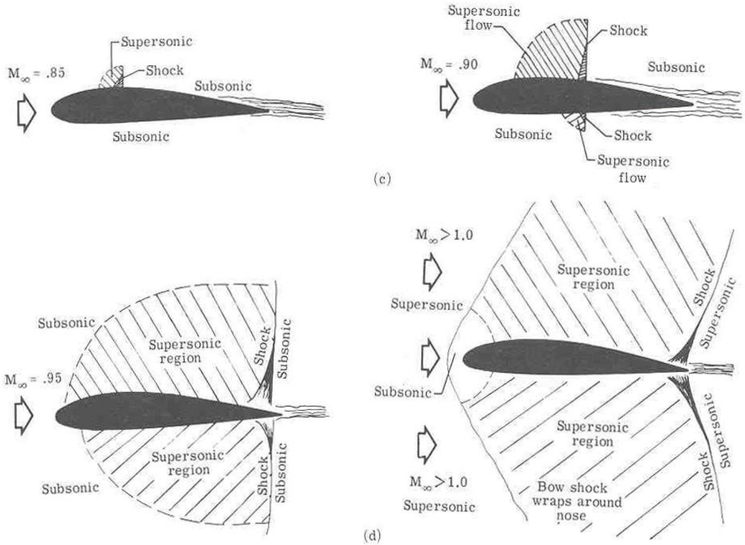




Airfoil Problems: Transonic flow past airfoils



Airfoil Problems: Transonic flow past airfoils



Nonlinear PDEs of Mixed Type and Airfoil Problems I

Steady Euler equations for Potential Flow:

$$\nabla \cdot (\rho(\nabla\varphi)\nabla\varphi) = 0, \qquad \mathbf{x} = (x, y) \in \mathbb{R}^2$$

or in the equivalent form:

$$\begin{cases} \nabla_{\mathbf{x}} \cdot (\rho(\mathbf{v})\mathbf{v}) = 0, \\ \nabla_{\mathbf{x}} \times \mathbf{v} = 0. \end{cases}$$

Flow velocity $\mathbf{v} := (u, v) = \nabla\varphi = (\varphi_x, \varphi_y)$, Flow speed $q = \sqrt{u^2 + v^2} = |\mathbf{v}|$.

For a γ-law gas, $p = p(\rho) = \rho^\gamma/\gamma, \gamma > 1$, is the normalized pressure. Then the Bernoulli's law is:

$$\rho = \rho(\mathbf{v}) := \left(1 - \frac{\gamma - 1}{2}|\mathbf{v}|^2\right)^{\frac{1}{\gamma-1}}.$$

Define $c = \sqrt{1 - \frac{\gamma-1}{2}q^2}$ (sonic speed), $q_{\rm cr} :\equiv \sqrt{\frac{2}{\gamma+1}}$ (critical speed).

We can rewrite Bernoulli's law in the form:

$$q^2 - q_{\rm cr}^2 = \frac{2}{\gamma+1}\left(q^2 - c^2\right).$$

Nonlinear PDEs of Mixed Type and Airfoil Problems II

$$(c^2 - u^2)\varphi_{xx} - 2uv\varphi_{xy} + (c^2 - v^2)\varphi_{yy} = 0.$$

The characteristic equation is:

$$(c^2 - u^2)\lambda^2 - 2uv\lambda + (c^2 - v^2) = 0$$

with eigenvalues:

$$\lambda = \frac{uv \pm c\sqrt{u^2 + v^2 - c^2}}{c^2 - u^2}.$$

Thus the equation is:

- Hyperbolic when $u^2 + v^2 > c^2$ (supersonic)
- Elliptic when $u^2 + v^2 < c^2$ (subsonic)
- Transition boundary: $u^2 + v^2 = c^2$ (sonic)

Notice that Bernoulli's law in the form: $q^2 - q_{\rm cr}^2 = \frac{2}{\gamma+1}\left(q^2 - c^2\right)$.

$\Longrightarrow$ Then the flow is subsonic (elliptic) when $q < q_{\rm cr}$,

sonic (degenerate state) when $q = q_{\rm cr}$,

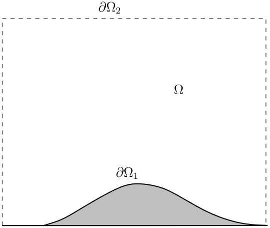
supersonic (hyperbolic) when $q > q_{\rm cr}$.

$$\nabla \cdot (\rho(\nabla\varphi)\nabla\varphi) = 0, \qquad \mathbf{x} \in \Omega,\ v_\infty = 0.$$

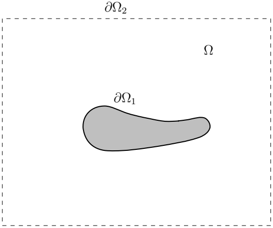
Obstacle Boundary $\partial\Omega_1$: Solid curve in (a); Solid closed curve in (b).
Far-field Boundary $\partial\Omega_2$: Dashed line segments in both (a) and (b).
Boundary conditions on the obstacle $\partial\Omega$:

$$\begin{cases} \nabla\varphi \cdot \mathbf{n} = 0 \qquad \text{on } \partial\Omega_1, \\ \text{Consistent far-field boundary conditions on } \partial\Omega_2, \end{cases}$$

where $\mathbf{n}$ is the unit normal pointing into the flow region on $\partial\Omega$.
In case (b), the circulation about the boundary $\partial\Omega_2$ is zero.



(a)



(b)

Nonlinear PDEs of Mixed Type and Airfoil Problems VI: Singularities

- **Stagnition**: Subsonic phase
- **Sonic states**: Transition between elliptic and hyperbolic phase

  Subsonic flow: Bers, Shiffman, Serrin, Finn, Gilbarg, Dong, $\cdots\cdots$

  Subsonic-sonic flow: Chen-Dafermos-Slemrod-Wang (CMP 2007)
  Compensated Compactness Framework
  Compactness and Existence

- **Cavitation**: Supersonic phase
- **Shock waves, rarefaction waves**: Supersonic-sonic phase
  Transonic phase

Morawetz Problem for Steady Potential Flow

Develop a compensated compactness framework such that a viscous approximate problem can be deigned in the form:

$$\begin{cases} \nabla_{\mathbf{x}} \cdot (\rho(\mathbf{v}^\varepsilon)\mathbf{v}^\varepsilon) = \varepsilon V_1^\varepsilon, \\ \nabla_{\mathbf{x}} \times \mathbf{v}^\varepsilon = \varepsilon V_2^\varepsilon, \end{cases}$$

with careful choice of the viscosity terms V_1^ε and V_2^ε, so that the corresponding viscous approximate solutions $\mathbf{v}^\varepsilon$ satisfy the compactness framework, which yields the convergence to a solution of the transonic flow problem.

Theorem (Morawetz 1985, 1995, 2004)

If viscous approximate solutions $\mathbf{v}^\varepsilon(\mathbf{x})$ *satisfy*

(i) *Uniformly away from Cavitation:* $|\mathbf{v}^\varepsilon(\mathbf{x})| \leq q_* < q_{\mathrm{cav}} < \infty$.

(ii) *Uniformly bounded flow-angle function:* $-\infty < \theta_* \leq \theta^\varepsilon(\mathbf{x}) \leq \theta^* < \infty$.

(iii) *Uniformly away from Stagnation:* $|\mathbf{v}^\varepsilon(\mathbf{x})| \geq \delta_0 > 0$.

Then there exists a subsequence of $\mathbf{v}^\varepsilon(\mathbf{x})$ *converging strongly to a solution of the transonic flow problem.*

Theorem (Chen-Slemrod-Wang: ARMA 2008, $\gamma \in (1,3)$)

- *Design a suitable viscous approximate problem.*
- *Assumptions* (i)–(ii) *can be removed.*

Morawetz Problem for Steady Potential Flow: $\gamma \geq 3$

Case: $\gamma \geq 3$: **cavitation can not be avoided, unlike the case $\gamma \in (1, 3)$.**
Case $\gamma = 3$ corresponds to case $\gamma = \frac{5}{3}$ for the isentropic Euler equations.

Entropy-Entropy Flux Pairs (Q_1, Q_2):

$$\nabla_{\mathbf{x}} \cdot (Q_1, Q_2) = -\Phi_\theta V_1 + \frac{q^2}{c^2 - q^2} \Phi_\rho V_2$$

with $\frac{c^2}{\rho^2 q^2} \Phi_{\theta\theta} + \left(\frac{q^2}{c^2 - q^2} \Phi_\rho \right)_\rho = 0.$

Then (Q_1, Q_2) is determined the generator H via the Loewner-Morawetz relation:

$$Q_1 = \rho q H_\mu \cos\theta - q H_\theta \sin\theta, \qquad Q_2 = \rho q H_\mu \sin\theta + q H_\theta \cos\theta.$$

Generator H satisfies the generalized **Tricomi-Keldysh equation**:

$$H_{\mu\mu} - \frac{M^2 - 1}{\rho^2} H_{\theta\theta} = 0,$$

where $M = \frac{q}{c}$ is the Mach number and μ is determined by $\mu'(\rho) = M^{-2}$.
The relation between the generator H and Φ is:

$$\rho H_{\rho\theta} - H_\theta = -\Phi_\theta, \qquad H_\mu + \frac{1}{\rho} H_{\theta\theta} = \frac{q^2}{c^2 - q^2} \Phi_\rho.$$

Morawetz Problem for Steady Potential Flow: $\gamma = 3$

Theorem (Chen-Giron-Schulz 2024: Compensated Compactness Framework)

Let $\{\mathbf{v}^\varepsilon\}_{\varepsilon>0} \subset L^\infty(\Omega)$ be a sequence of function satisfying

(i) **There exists $q^* \in (q_{\rm cr}, q_{\rm cav})$, indept of $\varepsilon > 0$, such that**

$$q_{\rm cr} < q_* \le |\mathbf{v}(\mathbf{x})| \le q_{\rm cav} \qquad a.e.\ \mathbf{x} \in \Omega.$$

(ii) **There exist $\theta_*, \theta^* \in \mathbb{R}$, indept of $\varepsilon > 0$, such that**

$$\theta_* \le \theta^\varepsilon(\mathbf{x}) \le \theta^* \qquad a.e.\ \mathbf{x} \in \Omega.$$

(iii) **For any entropy-entropy flux pair Q,**

$$\{\nabla_{\mathbf{x}} \cdot \mathbf{Q}(\mathbf{v}^\varepsilon)\}_{\varepsilon>0} \quad \text{is pre-compact in } H^{-1}_{\rm loc}.$$

$\Longrightarrow$ **there exists a subsequence of $\mathbf{v}^\varepsilon(\mathbf{x})$ converging in $L^p_{\rm loc}$ for for all $p \in [1, \infty)$.**

The choice of the viscosity terms V_1^ε and V_2^ε:

$$V_1^\varepsilon = \nabla_{\mathbf{x}} \cdot \big((1 - \frac{c(\rho(\mathbf{v}^\varepsilon))}{|\mathbf{v}^\varepsilon|^2})\nabla_{\mathbf{x}}\rho(\mathbf{v}^\varepsilon)\big),$$

$$V_2 = \Delta_{\mathbf{x}}\theta(\mathbf{v}^\varepsilon).$$

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(iii) For any entropy-entropy flux pair Q,

$$\{\nabla_{\mathbf{x}} \cdot \mathbf{Q}(\mathbf{v}^\varepsilon)\}_{\varepsilon>0} \quad \text{is pre-compact in } H^{-1}_{\rm loc}.$$

$\Rightarrow$ there exists a subsequence of $\mathbf{v}^\varepsilon(\mathbf{x})$ converging in $L^p_{\rm loc}$ for all $p \in [1, \infty)$.

- Conditions (i)–(iii) can be verified by the identification of the invariant regions for the viscous solutions for the supersonic incoming flow and the careful analysis of the behaviors of the entropy generator H as solutions of the Tricomi equations that are singular at cavitation.
- Compactness of exact solution sequences containing cavitation.
- Weak Continuity of the steady Euler equations with cavitation.
- Convergence of the viscous approximate solutions with cavitation $\cdots$

Weak Convergence and Weak Continuity

Recall: For a sequence $\mathbf{v}^\varepsilon : \Omega \to \mathbb{R}^m$ bounded in $L^\infty(\Omega)$, there exists a subsequence (still denoted) $\mathbf{v}^\varepsilon$ and a function $\mathbf{v} \in L^\infty(\Omega)$ such that $\mathbf{v}^\varepsilon \xrightarrow{*} \mathbf{v}$ in $L^\infty(\Omega)$:

$$\int_\Omega \mathbf{v}^\varepsilon \, g \, \mathrm{d}\mathbf{x} \to \int_\Omega \mathbf{v} \, g \, \mathrm{d}\mathbf{x} \qquad \text{as } \varepsilon \to 0,$$

for each $g \in L^1(\Omega)$.

For a continuous function $f \in C(\mathbb{R}^m)$, $f(\mathbf{v}^\varepsilon)$ is uniformly bounded in $L^\infty(\Omega; \mathbb{R}^m)$ so that

$$f(\mathbf{v}^\varepsilon) \xrightarrow{*} \bar{f}.$$

The question is:

$$\bar{f}(\mathbf{x}) = f(\mathbf{v}(\mathbf{x}))?$$

The answer is: NO, in general.

Example: Weak Convergence $\neq$ Weak Continuity

Consider the sequence: $u^\varepsilon = \cos(\frac{x}{\varepsilon})$.
For any test function $\phi \in C_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \cos(\frac{x}{\varepsilon})\,\phi(x)\,\mathrm{d}x = -\varepsilon \int_{\mathbb{R}} \sin(\frac{x}{\varepsilon})\,\phi'(x)\,\mathrm{d}x \to 0 \qquad \text{as } \varepsilon \to 0,$$

since $\int_{\mathbb{R}} \sin(\frac{x}{\varepsilon})\,\phi'(x)\,\mathrm{d}x$ is bounded.

However, for

$$(u^\varepsilon)^2 = \cos^2(\frac{x}{\varepsilon}) = \frac{1}{2}\big(1 + \cos(\frac{2x}{\varepsilon})\big),$$

we have

$$\int_{\mathbb{R}} \cos^2(\frac{x}{\varepsilon})\,\phi(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \cos(\frac{2x}{\varepsilon})\,\phi(x)\,\mathrm{d}x \to \frac{1}{2}\int_{\mathbb{R}} \phi(x)\,\mathrm{d}x.$$

Therefore,

$$u^\varepsilon(x) = \cos(\frac{x}{\varepsilon}) \overset{*}{\rightharpoonup} u = 0,$$

but

$$(u^\varepsilon)^2(x) = \cos^2(\frac{x}{\varepsilon}) \overset{*}{\rightharpoonup} \frac{1}{2} \neq u^2 = 0.$$

Lemma (Classical Div-Curl Lemma: Murat-Tartar)

Let $\Omega \subset \mathbb{R}^d, d \geq 2,$ *be open bounded. Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Assume that, for* $\varepsilon > 0$*, two fields* $\mathbf{u}^\varepsilon \in L^p(\Omega; \mathbb{R}^d)$, $\mathbf{v}^\varepsilon \in L^q(\Omega; \mathbb{R}^d)$ *satisfy the following*:

1. $\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}$ *weakly in* $L^p(\Omega; \mathbb{R}^d)$ *as* $\varepsilon \to 0$;
2. $\mathbf{v}^\varepsilon \rightharpoonup \mathbf{v}$ *weakly in* $L^q(\Omega; \mathbb{R}^d)$ *as* $\varepsilon \to 0$;
3. $\operatorname{div} \mathbf{u}^\varepsilon$ *are confined in a compact subset of* $W^{-1,p}_{\mathrm{loc}}(\Omega; \mathbb{R})$;
4. $\operatorname{curl} \mathbf{v}^\varepsilon$ *are confined in a compact subset of* $W^{-1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^{d\times d})$.

Then the scalar product of $\mathbf{u}^\varepsilon$ *and* $\mathbf{v}^\varepsilon$ *are weakly continuous:*

$$\mathbf{u}^\varepsilon \cdot \mathbf{v}^\varepsilon \longrightarrow \mathbf{u} \cdot \mathbf{v} \qquad \text{in the sense of distributions.}$$

***Various variations of this lemma for different applications/purposes**:

Robbin-Rogers-Temple (1987)

Kozono-Yanagisawa (2009)

Cont-Dolzmann-Müller (2011)

. . .

Compensated Compactness and Weak/Strong Convergence

Let $\mathbf{v}^\varepsilon : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^m$ be a sequence of measurable functions satisfying:

- $\mathbf{v}^\varepsilon$ is uniformly bounded: $\exists$ bounded set $K \in \mathbb{R}^m$ such that $\mathbf{v}^\varepsilon(\mathbf{x}) \in K$ a.e.
- For two vector functions $\mathbf{Q}^j = (Q_1^j, Q_2^j)$, $j = 1, 2$,

$$\{\nabla_\mathbf{x} \cdot \mathbf{Q}_j(\mathbf{v}^\varepsilon)\}_{\varepsilon>0} \quad \text{is pre-compact in } H^{-1}_{\mathrm{loc}}.$$

$\Longrightarrow$ There exists a subsequence (still denoted) $\mathbf{v}^\varepsilon$ and a family of Young measures

$$\nu_\mathbf{x} : \Omega \to \mathrm{Prob.}(\mathbb{R}^m) \qquad \text{with} \quad \mathrm{supp}\,\nu_\mathbf{x} \subset \bar{K},$$

such that

(i) For any continuous function $f(\,\cdot\,)$, the weak-star limit has the following Young measure representation:

$$w^* - \lim f(\mathbf{v}^\varepsilon) = \langle \nu_\mathbf{x}(\lambda), f(\boldsymbol{\lambda}) \rangle = \int_{\mathbb{R}^m} f(\boldsymbol{\lambda}) \mathrm{d}\nu_\mathbf{x}(\lambda),$$

(ii) The Young measure ν_x is governed by the following commutativity relation with respect to the vector functions $\mathbf{Q}^j$, $j = 1, 2$:

$$\langle \nu_\mathbf{x}, Q_1^1 Q_2^2 - Q_2^1 Q_1^2 \rangle = \langle \nu_\mathbf{x}, Q_1^1 \rangle \langle \nu_\mathbf{x}, Q_2^2 \rangle - \langle \nu_\mathbf{x}, Q_2^1 \rangle \langle \nu_\mathbf{x}, Q_1^2 \rangle.$$

(iii) $\mathbf{v}^\varepsilon(\mathbf{x}) \to \mathbf{v}(x)$ strongly if and only if $\nu_\mathbf{x}$ is a Dirac mass:

$$\nu_x = \delta_{\mathbf{v}(\mathbf{x})} \qquad a.e. \quad \text{in } \Omega.$$

Reduction Problem:

Let the Young measure (probability measure) $\nu_{\mathbf{x}}(\boldsymbol{\lambda})$ be governed by

$$\langle \nu_{\mathbf{x}}(\boldsymbol{\lambda}), Q_1^1(\boldsymbol{\lambda})Q_2^2(\boldsymbol{\lambda}) - Q_2^1(\boldsymbol{\lambda})Q_1^2(\boldsymbol{\lambda})\rangle$$
$$= \langle \nu_{\mathbf{x}}(\boldsymbol{\lambda}), Q_1^1(\boldsymbol{\lambda})\rangle\langle \nu_{\mathbf{x}}(\boldsymbol{\lambda}), Q_2^2(\boldsymbol{\lambda})\rangle - \langle \nu_{\mathbf{x}}(\boldsymbol{\lambda}), Q_2^1(\boldsymbol{\lambda})\rangle\langle \nu_{\mathbf{x}}(\boldsymbol{\lambda}), Q_1^2(\boldsymbol{\lambda})\rangle.$$

for entropy pairs (Q_1^j, Q_2^j), $j = 1, 2$, is determined by the generators H^j via the Loewner-Morawetz relation:

$$Q_1^j = \rho q H_\mu^j \cos\theta - q H_\theta^j \sin\theta, \qquad Q_2 = \rho q H_\mu^j \sin\theta + q H_\theta^j \cos\theta.$$

Generators H^j are governed by the **Tricomi-Keldysh-type equation**:

$$H_{\mu\mu} - \frac{M^2(\mu) - 1}{\rho^2(\mu)} H_{\theta\theta} = 0, \qquad M = \frac{q}{c} \text{ – Mach number.}$$

Issue: **Is $\nu_{\mathbf{x}}$ a Dirac measure?**

$\Longrightarrow$ **Compactness of $\mathrm{v}^\varepsilon(\mathrm{x})$ in L^1.**

*Div-Curl Lemma: Murat (1978), Tartar (1979) , $\cdots\cdots$

*Young Measure Rep.: Tartar (1979), Ball (1989), Alberti-Müller (2001), $\cdots\cdots$

Nonlinear PDEs of Mixed Type and Other Steady Transonic Problems in Fluid Dynamics

Richard Courant & Kurt Friedrichs: **Supersonic Flow and Shock Waves**
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