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ON EXISTENCE OF SOLUTION FOR A THERMOELASTIC BEAM MODEL

C. N. CUNHA & M. R. CLARK

[1]Universidade Federal do Delta do Parnaíba, Parnaíba, Brazil,
[2] Universidade Federal do Piauí, Teresina, Brazil

Abstract

This paper addresses the Cauchy problem associated with the nonlinear thermoelastic beam featuring thermal dissipation. We examine the problem in an open domain, which may be either bounded or unbounded. The existence of a solution is derived through the diagonalization theorem for self-adjoint operators.

1 Introduction

This work addresses the Cauchy problem associated with the nonlinear thermoelastic beam featuring thermal dissipation. The model is written as follows

$$\begin{aligned} &u'' + \Delta^2 u - M\left(\int_\Omega |\nabla u(x,\cdot)|^2 dx\right)\Delta u + \theta = f, \\ &\theta' - \Delta\theta + u' = g, \end{aligned} \tag{1}$$

where Δ denote the Laplace operator, ∇ is the gradient operator and M is a positive real function defined on $[0, +\infty)$. The system (1) describes the vibrations of an extensible thermoelastic beam, being composed of a hyperbolic equation (elastic behavior) and a parabolic equation (thermal behavior of the medium). Physically, $u = u(x,t)$ represents the deflection at a point x of the beam at instant t (from the rest configuration) and $\theta = \theta(x,t)$ the variation of the temperature (relative to a reference value), while $f = f(x,t)$ and $g = g(x,t)$ are, respectively, the lateral load distribution and an external heat source (see [5]). Concerning the additional damping in the parabolic equation, it has been observed in the literature which plays a role in dissipating the energy of the system while maintaining the amplitude of elastic and thermal oscillations. This is crucial to ensure the asymptotic stability of the solution (see [4]). Furthermore, in [2], which considered one of the pioneers in the study of thermoelasticity, the author emphasizes that the thermoelastic model is suitable for investigating the vibrations of certain elastic bodies. Consequently, it can be concluded that the aforementioned model is physically viable.

In this paper, we study the system (1) without using the compactness method. Motivated by the papers [6] and [7], we will consider the abstract formulation of system (1)

$$(P.1) \qquad \begin{aligned} &u'' + A^2 u + M(\cdot, |A^{1/2}u|^2)Au + \theta = f, \\ &\theta' + A\theta + u' = g, \\ &u(0) = u_0,\ u'(0) = u_1 \text{ and } \theta(0) = \theta_0, \end{aligned}$$

where A is a positive self-adjoint operator in separable Hilbert space H, with inner product $(\cdot,\cdot)$ and the norm $|\cdot|$, and M is a positive real function. We prove the existence and uniqueness of the solution for the problem (P.1) under appropriate assumptions on M, by making use of the diagonalization theorem. This strategy offers the advantage of being applicable in both bounded and unbounded domains.

2 Main Results

Let us fix a real number $T > 0$, a operator A as above and the real function M. Suppose that

$$M \in C^1([0,T] \times [0,+\infty); \mathbb{R}), \tag{2}$$

$$M(t,\lambda) \geq m_0 > 0, \ \forall \ (t,\lambda) \in [0,T] \times [0,+\infty), \tag{3}$$

$$\frac{\partial M}{\partial t}(t,\lambda) \leq 0, \ \forall \ (t,\lambda) \in [0,T] \times [0,+\infty), \tag{4}$$

$$\{u_0, u_1, \theta_0\} \in D(A) \times H \times D(A^{1/2}), \tag{5}$$

$$\{f, g\} \in \left[L^2(0,T;H)\right]^2. \tag{6}$$

Then, our main result is:

Theorem 2.1. *Suppose (2)-(6) holds. Then, there exist exactly one pair $u, \theta : [0,T] \longrightarrow H$ of vector functions satisfying*

$$u \in L^\infty(0,T;D(A)), \tag{7}$$

$$u' \in L^\infty(0,T;H), \tag{8}$$

$$\theta \in L^2(0,T;D(A^{1/2})), \tag{9}$$

$$\begin{aligned} \frac{d}{dt}(u'(t),z) + (Au(t),Az) + M(t,|A^{1/2}u(t)|^2)(A^{1/2}u(t),A^{1/2}z) \\ +(\theta(t),z) = (f(t),z), \ \forall z \in D(A), \end{aligned} \tag{10}$$

in the sense of $L^2(0,T)$,

$$\frac{d}{dt}(\theta(t),z) + (A^{1/2}\theta(t),A^{1/2}z) + (u'(t),z) = (g(t),z), \ \forall z \in D(A^{1/2}), \tag{11}$$

in the sense of $L^2(0,T)$,

$$u(0) = u_0, \ u'(0) = u_1 \ e \ \theta(0) = \theta_0. \tag{12}$$

The pair (u,θ) above is said to be a local weak solution of system (P.1). We prove Theorem 2.1 in steps. We perturb the system (P.1) and we use of the unitary operator $\mathcal{U}$, defined by Diagonalization Theorem (cf. [3]). Finally, the solution is obtained through Arzelá-Áscoli Theorem, using an appropriate topology.

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NUMERICAL ANALYSIS OF THE MAXWELL-CATTANEO-VERNOTTE NONLINEAR MODEL

A. J. A. RAMOS[1], A. D. S. CAMPELO[1], M. M. FREITAS[1] & R. KOVÁCS[2]

[1] Federal University of Pará, Salinópolis, PA, Brazil, ramos@ufpa.br[1], campelo@ufpa.br[2], mirelson@ufpa.br[3]

[2] Budapest University of Technology and Economics, Budapest, Hungary, kovacs.robert@energia.bme.hu

Abstract

In the literature, one can find numerous modifications of Fourier's law, the first of which is the Maxwell–Cattaneo–Vernotte heat equation. Although this model has been known for decades and successfully used to model low-temperature damped heat wave propagation, its nonlinear properties are rarely investigated. This work presents the functional relationship between the transport coefficients and the consequences of their temperature dependence, particularly focusing on thermal conductivity. Furthermore, we introduce a particular implicit numerical scheme to solve such nonlinear heat equations reliably, free from artificial numerical errors.

1 Introduction

In recent years, numerous heat conduction models have been developed to provide a more efficient modeling tool for complex problems related to wave propagation under low-temperature conditions [1], in rarefied media [2, 3], in nanosystems [4, 5], or over-diffusion in complex heterogeneous material structures [6].

Here we derive a Maxwell-Cattaneo-Vernotte nonlinear model in which the temperature dependence is included in the thermal conductivity, *i.e.*

$$\rho c T_t + q_x = 0 \quad \text{in} \quad (0,\ell)\times(0,\infty), \tag{1}$$

$$\tau q_t + q + aT\,T_x + \lambda T_x = 0 \quad \text{in} \quad (0,\ell)\times(0,\infty), \tag{2}$$

where q and T are the heat flux and temperature, τ, ρ, c, a and λ are physical constants. We consider two types of boundary conditions:

$$\textbf{Boundary type I:} \quad \begin{cases} q(0,t) = 0, & \text{for all} \quad t \geq 0, \\ q(\ell,t) = 0, & \text{for all} \quad t \geq 0, \end{cases} \tag{3}$$

$$\textbf{Boundary type II:} \quad \begin{cases} q(0,t) = \begin{cases} 1-\cos\left(2\pi t/t_p\right), & \text{if} \quad 0 < t \leq t_p, \quad t_p > 0, \\ 0, \quad \text{if} \quad t > t_p, \end{cases} \\ q(\ell,t) = 0, \quad \text{for all} \quad t \geq 0, \end{cases} \tag{4}$$

for which we also assign two types of initial conditions:

$$\textbf{Initial condition I:} \quad T(x,0) = T_0(x), \quad q(x,0) = q_0(x), \quad x \in (0,\ell), \tag{5}$$

$$\textbf{Initial condition II:} \quad T(x,0) = T_0, \quad q(x,0) = q_0 \equiv 0, \quad x \in (0,\ell). \tag{6}$$

2 Main Results

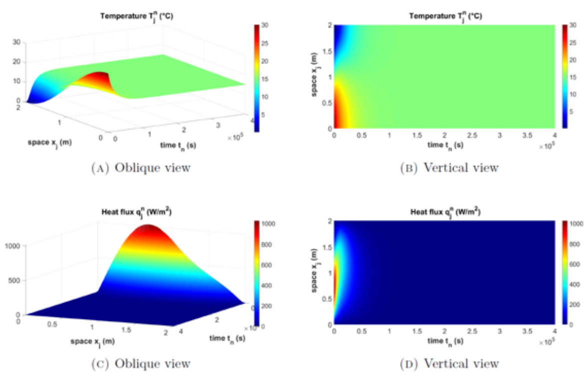
Our main result is a numerical linearization method in finite differences given by

$$\textbf{Boundary type I:}\quad \begin{cases} \Phi + \dfrac{r}{\rho c}\mathbf{A}\Psi = -\dfrac{r}{\rho c}\mathbf{A}\mathbb{Q}^{n-1}, \\ \Psi + \dfrac{1}{2(\tau+\Delta t)}\mathbf{B}\Phi = -\dfrac{r}{2(\tau+\Delta t)}\mathbf{C}\Big(a\mathbf{D}+2\lambda\mathbf{I}_{J+1}\Big)\mathbb{T}^{n-1} - \dfrac{\Delta t}{(\tau+\Delta t)}\mathbb{Q}^{n-1}, \end{cases} \tag{7}$$

where Φ is a vector related to T and Ψ, $\mathbb{Q}^{n-1}$ are vectors related to q. Furthermore, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{I}_{J+1}$ are matrices.

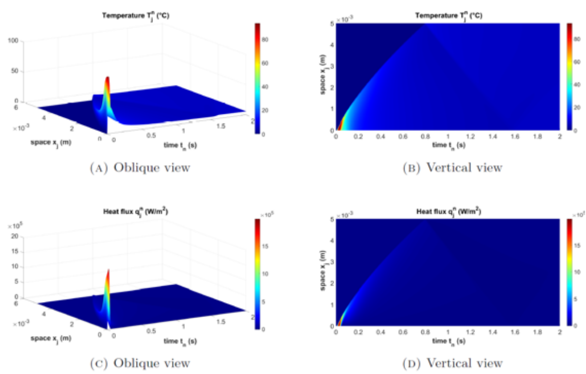
$$\textbf{Boundary type II:}\quad \begin{cases} \Phi + \dfrac{r}{\rho c}\mathbf{A}\Psi = \begin{cases} -\frac{r}{\rho c}\mathbf{A}\mathbb{Q}^{n-1} + \frac{r}{\rho c}\Big(1-\cos\big(2\pi t_n/t_p\big)\Big)\mathbf{L}, & \text{if} \quad 0 < n \leq p, \quad p \in \mathbb{N} \\ -\frac{r}{\rho c}\mathbf{A}\mathbb{Q}^{n-1}, & \text{if} \quad n > p, \end{cases} \\ \Psi + \dfrac{1}{2(\tau+\Delta t)}\mathbf{B}\Phi = -\dfrac{r}{2(\tau+\Delta t)}\mathbf{C}\Big(a\mathbf{D}+2\lambda\mathbf{I}_{J+1}\Big)\mathbb{T}^{n-1} - \dfrac{\Delta t}{(\tau+\Delta t)}\mathbb{Q}^{n-1}, \end{cases} \tag{8}$$

where $\mathbb{T}^{n-1}$ is a vector related to T and $\mathbf{L}$ is a matrix. Below, we implement the same parameters and solve the difference equations for types **I** and **II** (see Fig. 1 and 2).



(A) Oblique view (B) Vertical view (C) Oblique view (D) Vertical view

Figure 1: Boundary type I



(A) Oblique view (B) Vertical view (C) Oblique view (D) Vertical view

Figure 2: Boundary type II

We note that temperature-dependent thermal conductivity distorts the symmetric evolution. The present asymmetry indicates how the thermal conductivity depends on the temperature (increasing or decreasing), also proposing a method to observe this effect experimentally.

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EXISTENCE OF A POSITIVE SOLUTION TO A SECOND-ORDER NONLINEAR PROBLEM WITH MIXED BOUNDARY CONDITIONS: A SUPERLINEAR CASE

ADRIANO PEIXOTO

Universidade de São Paulo, IME, SP, Brazil, alcp@ime.usp.br

Abstract

In this work, we present the existence of a positive solution to a second-order nonlinear problem with mixed boundary conditions. The proofs of the main results are based on the Mawhin's coincidence degree.

1 Introduction

The goal is to prove the existence of positive solutions for the problem

$$\begin{cases} u'' + b(t)g(u) = 0, \ 0 < t < T \\ u(0) = u(T) \text{ and } u'(0) = 0, \end{cases} \qquad (\mathcal{E})$$

where $g : [0, +\infty) \to [0, +\infty)$ is a continuous function such that

(g_1) $\qquad g(0) = 0, \quad g(s) > 0$ for $s > 0$.

The weight coefficient $b : [0, T] \to \mathbb{R}$ is a L^1-function such that

(b_1) there exists $\delta > 0$ such that $b(t)$ is essentially negative on $[0, \delta]$ and also on $[T - \delta, T]$;

(b_2) there exist $m \geq 1$ intervals $I_1, \ldots, I_m$, closed and pairwise disjoint, such that

$$b(t) \geq 0, \text{ for a.e. } t \in I_i, \text{ with } b(t) \not\equiv 0 \text{ on } I_i \quad (i = 1, \ldots, m);$$

$$b(t) \leq 0, \text{ for a.e. } t \in [0, T] \setminus \bigcup_{i=1}^{m} I_i;$$

(b_3) $\qquad \int_0^s b(t)dt < 0$ for all $0 < s < T$.

Let $\lambda_1^i, i = 1, \ldots, m$, be the first eigenvalue of the eigenvalue problem

$$\varphi'' + \lambda b(t)\varphi = 0, \quad \varphi|_{\partial I_i} = 0.$$

A function $g : [0, +\infty) \to [0, +\infty)$ satisfying (g_1) is *regularly oscillating* at zero if

$$\lim_{\substack{s \to 0^+ \\ \omega \to 1}} \frac{g(\omega s)}{g(s)} = 1.$$

Before proving the existence of a positive solution to problem $(\mathcal{E})$, we study the more general problem

$$\begin{cases} u'' + f(t, u, u') = 0, \quad 0 < t < T \\ u(0) = u(T) \text{ and } u'(0) = 0 \end{cases} \qquad (1)$$

where $f : [0, T] \times [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ be an L^p-Carathéodory function, for some $1 \leq p \leq \infty$, satisfying certain conditions named $(f_1), (f_2)$ and (f_3).

2 Main Results

Theorem 2.1. *Assume* (f_1)*,* (f_2)*, and* (f_3)*, and suppose that there exist two constants* $r, R > 0$*, with* $r \neq R$*, such that the following hypotheses are true.*

(H_1) *The condition are satisfied:*

$$\int_0^T \left(\int_0^s f(t, r, 0) dt \right) ds < 0.$$

are satisfied. Moreover, any solution $u(t)$ *of the problem*

$$\begin{cases} u'' + \vartheta f(t, u, u') = 0, \ \ 0 < t < T \\ u(0) = u(T) \ \text{and} \ u'(0) = 0, \end{cases} \tag{2}$$

for $0 < \vartheta \leq 1$*, such that* $u(t) > 0$ *in* $[0, T]$*, satisfies* $\|u\|_\infty \neq r$*.*

(H_2) *There exist a non-negative function* $v \in L^p([0, T], \mathbb{R})$ *with* $v \not\equiv 0$ *and a constant* $\alpha_0 > 0$*, such that every solution* $u(t) \geq 0$ *of the problem*

$$\begin{cases} u'' + f(t, u, u') + \alpha v(t) = 0, \ \ 0 < t < T \\ u(0) = u(T) \ \text{and} \ u'(0) = 0, \end{cases} \tag{3}$$

for $\alpha \in [0, \alpha_0]$*, satisfies* $\|u\|_\infty \neq R$*.*

(H_3) *There are no solutions* $u(t)$ *of (3) for* $\alpha = \alpha_0$ *with* $0 \leq u(t) \leq R$*, for every* $t \in [0, T]$*.*

Then the problem (1) has at least one positive solution $u(t)$ *with*

$$\min\{r, R\} < \max_{t \in [0,T]} u(t) < \max\{r, R\}.$$

Proof. The proof is given by a topological approach based on the Mawhin's coincidence degree. Furthermore, to ensure that the found solution is positive, we employ a maximum principle. ■

Theorem 2.2. *Let* $g(s)$ *and* $b(t)$ *be as in the introduction. Suppose also that* $g(s)$ *is regularly oscillating at zero and satisfies*

$$(g_2) \qquad \lim_{s \to 0^+} \frac{g(s)}{s} = 0 \qquad \text{and} \qquad g_\infty := \liminf_{s \to +\infty} \frac{g(s)}{s} > \max_{i=1,\ldots,m} \lambda_1^i.$$

Then problem $(\mathcal{E})$ *has at least one positive solution.*

Proof. The proof is based on showing that the hypotheses about $g(u)$ and $b(t)$ allow us to apply Theorem 2.1. ■

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CRITICAL DOUBLE PHASE EQUATIONS IN $\mathbb{R}^N$ WITH LOGARITHMIC NONLINEARITIES

A. BAHROUNI[1], A. FISCELLA[2], & P. WINKERT[3]

[1]Mathematics Department, Faculty of Sciences, University of Monastir, 5019 Monastir, Tunisia, bahrounianouar@yahoo.fr
[2]Departamento de Matemática, Universidade Estadual de Campinas, IMECC, Rua Sérgio Buarque de Holanda 651, 13083–859 Campinas SP, Brazil, fiscella@unicamp.br
[3]Technische Universität Berlin, Institut für Mathematik, Straße des 17. Juni 136, 10623 Berlin, Germany, winkert@math.tu-berlin.de

Abstract

In this talk, we discuss about double phase equations set in $\mathbb{R}^N$ involving critical Sobolev terms and logarithmic nonlinearities. More precisely, our equations are driven by the so-called *double phase operator* given by

$$u \in W^{1,\mathcal{H}}(\mathbb{R}^N) \mapsto \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{p-2}\nabla u\right),$$

set on an appropriate Musielak-Orlicz-Sobolev space $W^{1,\mathcal{H}}(\mathbb{R}^N)$, with $1 < p < q < \infty$ and $\mu \in L^\infty(\mathbb{R}^N)$ such that $\mu(x) \geq 0$ a.e. in $\mathbb{R}^N$.

By variational methods, we provide different existence results for our equations. The main difficulty arises from the presence of logarithmic nonlinearity, which is sign-changing, combined with a double lack of compactness, due to the free action of translation group in $\mathbb{R}^N$ and the critical Sobolev nonlinearity. Furthermore, we have to deal with Luxemburg type norm of $W^{1,\mathcal{H}}(\mathbb{R}^N)$, which complicates even the study of geometry for the energy functional.

Our results are new even in the classical p-Laplacian case, that is when $\mu \equiv 0$.

1 Introduction

In this talk, we present the following equation in $\mathbb{R}^N$

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) + |u|^{p-2}u + \mu(x)|u|^{q-2}u = K_1(x)|u|^{p^*-2}u + \lambda K_2(x)|u|^{r-2}u\log(|u|) \\ + \gamma K_3(x)|u|^{\beta-2}u, \end{aligned} \tag{1}$$

where the main operator on the left-hand side is the so-called double phase operator satisfying the structural assumption:

(H_1) *$1 < p < q < N$, $q < p^* = \frac{Np}{N-p}$ and $\mu\colon \mathbb{R}^N \to [0,\infty)$ is Lipschitz continuous such that $\mu \in L^\infty(\mathbb{R}^N)$.*

Here, we consider parameters λ, $\gamma > 0$ and exponents $r \in [q, p^*)$, $\beta \in (1, p^*)$. Concerning the functions K_1, K_2, $K_3 : \mathbb{R}^N \to \mathbb{R}$, we assume the following conditions:

(H_2) *$K_1 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $K_1(x) > 0$ for all $x \in \mathbb{R}^N$ and if $\{A_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ is a sequence of Borel sets such that the Lebesgue measure $|A_n| \leq R$ for all $n \in \mathbb{N}$ and some $R > 0$, then*

$$\lim_{n\to\infty} \int_{A_n \cap B_\rho^c(0)} K_1(x)\,\mathrm{d}x = 0,$$

for some $\rho > 0$.

(H_3) *$K_2 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $0 < K_2 < K_1$ on $\mathbb{R}^N$, and $K_3 \in L^\infty(\mathbb{R}^N)$ with $0 < K_3 < K_1$ on $\mathbb{R}^N$.*

(H_4) *there exists* $\sigma \in (q, \beta)$ *such that*

$$K_2(x) \leq \frac{e\,(r-\beta)\,r\,(\beta-\sigma)}{\beta(r-\sigma)} K_3(x), \quad \text{for any } x \in \mathbb{R}^N.$$

$(\widetilde{H_3})$ $K_2 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *with* $0 < K_2 < K_1$ *on* $\mathbb{R}^N$, *and* $K_3 \in L^1(\mathbb{R}^N) \cap L^{\frac{q}{q-\beta}}(\mathbb{R}^N)$ *with* $0 < K_3 < K_1$ *on* $\mathbb{R}^N$.

2 Main Results

In this talk, we study equation (1) under two main cases: the superlinear case, with $1 < p < q < \beta < r < p^*$; the linear case, with $1 < \beta < p < r = q < p^*$. By applying different variational methods, we provide existence results for (1), proved in [1] and stated below.

Theorem 2.1. *Let* $\gamma = \lambda$. *Let* (H_1) – (H_4) *be satisfied and let* $1 < p < q < \beta < r < p^*$. *Then, there exists* $\lambda^* > 0$ *such that, if* $\lambda \geq \lambda^*$, *equation* (1) *admits at least one nontrivial weak solution.*

Theorem 2.2. *Let* $\gamma = 1$. *Let* (H_1) – (H_2) *and* $(\widetilde{H_3})$ *be satisfied and let* $1 < \beta < p < r = q < p^*$. *Then, for any* $\lambda > 0$, *there exists* $k_\lambda > 0$ *such that if*

$$\max\left\{\|K_3\|_1, \|K_3\|_{\frac{q}{q-\beta}}\right\} < k_\lambda, \tag{2}$$

equation (1) *admits at least one nontrivial weak solution.*

The weak solution of Theorem 2.1 is obtained by a mountain pass argument. In order to overcome the double lack of compactness of (1), arising from the free action of translation group in $\mathbb{R}^N$ and the critical Sobolev nonlinearity, we exploit a tricky step analysis for the critical mountain pass level c_λ. In particular, we first prove the asymptotic property

$$\lim_{\lambda \to \infty} c_\lambda = 0. \tag{3}$$

In this direction, the restrictive assumption (H_4) is crucial to prove (3) and to prove the validity of the Palais-Smale compactness condition at level c_λ.

While, the proof Theorem 2.2 is based on a suitable minimization argument. For this, we need to control the β-exponent nonlinearity in (1) with hypothesis (2). However, it strongly forces the restriction $\beta \in (1, p)$. It is still an open problem the linear case $r = q$ with $\beta \in [p, q)$.

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APPLICATION OF THE SEMIGROUP THEORY TO A COMBUSTION PROBLEM IN A MULTI-LAYER POROUS MEDIUM

E. ALARCON[1], M. BATISTA[2], A. CUNHA[1], J. DA MOTA[1], & R. SANTOS[1]

[1]Universidade Federal de Goiás, arbietoalarcon@ufg.br, alysson@ufg.br, jesusdamota@gmail.com, rasantos@ufg.br
[2]Instituto Federal de Goiás, marcosroberto1995@gmail.com

Abstract

This study proved that the Cauchy problem for a one-dimensional reaction-diffusion-convection system is locally and globally well-posed in $\mathtt{H}^2(\mathbb{R})$. The system modeled a gasless combustion front through a multi-layer porous medium when the fuel concentration in each layer was a known function. More information about the physical modelling can be found in [2]. Here, we consider the continuity of the solution regarding the initial data and parameters, unlike the current study. This proof uses a novel approach to combustion problems in porous media. We follow the abstract semigroups theory of operators in the Hilbert space and the well-known Kato's theory for a well-posed associated initial value problem. See also [3].

1 Introduction

We study the following initial value problem is well-posed in $\mathtt{H}^2(\mathbb{R})$,

$$\begin{cases} u_t + L(t)u = f(x,t,u), \;\; x \in \mathbb{R}, \;\; t > 0, \\ u(x,0) = \phi(x). \end{cases} \tag{1}$$

Here, $u = (u_1, \dots, u_n)$ is an unknown vector of temperatures $\phi = (\phi_1, \dots, \phi_n)$ is the given vector of the initial temperatures, and $L(t)$ is the partial differential operator defined by

$$L(t)u = \big(L_1(t)u_1, \dots, L_n(t)u_n\big), \tag{2}$$

where

$$L_i(t)u_i := -\alpha_i(x,t)\,\partial_x^2 u_i + \beta_i(x,t)\,\partial_x u_i, \;\; i = 1, \dots, n, \tag{3}$$

$$\alpha_i(x,t) = \frac{\lambda_i(x)}{a_i(x) + b_i(x)\,y_i(x,\,t)}, \;\; \beta_i(x,t) = \frac{c_i(x)}{a_i(x) + b_i(x)\,y_i(x,\,t)}\,. \tag{4}$$

Functions a_i, b_i, c_i, and λ_i for $i = 1, \dots, n$ are defined depending on the physical parameters (see [2]), which are known functions of the spatial variable x. The combustion reaction rate, heat transfer between two adjacent layers, and heat loss to the external medium are all included in the source function $f = (f_1, \dots, f_n)$, the components of which are defined by

$$
\begin{aligned}
f_1(x,t,u) &= \frac{-(c_1)_x\, u_1}{a_1 + b_1 y_1} + \frac{(K_1 b_1 u_1 + d_1) y_1\, g(u_1)}{a_1 + b_1 y_1} + \frac{q_1(u_2 - u_1)}{a_1 + b_1 y_1} \\
&\quad - \frac{\overline{q}_1(u_1 - u_e)}{a_1 + b_1 y_1}\,, \\
f_i(x,t,u) &= \frac{-(c_i)_x\, u_i}{a_i + b_i y_i} + \frac{(K_i b_i u_i + d_i) y_i\, g(u_i)}{a_i + b_i y_i} - \frac{q_{i-1}(u_i - u_{i-1})}{a_i + b_i y_i} \\
&\quad + \frac{q_i(u_{i+1} - u_i)}{a_i + b_i y_i}, \quad i = 2, \ldots, n-1\,, \\
f_n(x,t,u) &= \frac{-(c_n)_x\, u_n}{a_n + b_n y_n} + \frac{(K_n b_n u_n + d_n) y_n\, g(u_n)}{a_n + b_n y_n} - \frac{q_{n-1}(u_n - u_{n-1})}{a_n + b_n y_n} \\
&\quad - \frac{\overline{q}_2(u_n - u_e)}{a_n + b_n y_n}\,,
\end{aligned}
\tag{5}
$$

where d_i is also a known function of the spatial variable x, and g is a function related to the Arrhenius law given by

$$
g(\theta) = \begin{cases} e^{-\frac{E}{\theta}}, & \text{se } \theta > 0 \\ 0, & \text{se } \theta \le 0\,. \end{cases}
\tag{6}
$$

The other quantities K_i, q_i, $\overline{q}_1$, $\overline{q}_2$, E are non-negative parameters, as defined in [2], and u_e denotes the temperature of the external environment, which is constant.

2 Main Results

Theorem 2.1 (Local solution)**.** *Assume that hypotheses (Hy6) and (Hy7) given in Section 3 (see [1]) are satisfied. If $\phi = (\phi_1, \ldots, \phi_n) \in \mathtt{H}^2(\mathbb{R})^n$, then the initial value problem (1) has a unique local solution. Each component of this solution is given in the following integral form:*

$$
u_i(t) = U_i(t,\, 0)\phi_i + \int_0^t U_i(t,\, \tau)\, f_i\big(\tau,\, u(\tau)\big)\, d\tau,
\tag{7}
$$

$t \in [0, T]$ for some $T > 0$, where U_i is the evolution propagation operator associated with $L_i(t)$.

Theorem 2.2 (Global solution)**.** *We assume that the hypothesis (Hy8) given in Section 3 (see [1]) is satisfied. If $\phi = (\phi_1, \ldots, \phi_n) \in \mathtt{H}^2(\mathbb{R})^n$, then the initial value problem (1) has a unique global solution. Each component of this solution is given in the integral form, as in (7), for any $T > 0$.*

Theorem 2.3 (Continuous dependence)**.** *Let us assume the same hypotheses as in Theorem 2.1. Then, the function that maps the initial data and the parameters into the solution given by this theorem is continuous in the $\mathtt{H}^2(\mathbb{R})^n$-norm. Similarly, let us assume the same hypotheses as in Theorem 2.1; then, the analogous result holds.*

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EXISTENCE AND STABILITY OF PULLBACK EXPONENTIAL ATTRACTORS FOR A NONAUTONOMOUS SEMILINEAR EVOLUTION EQUATION OF SECOND ORDER

V. T. AZEVEDO[1], E. M. BONOTTO[2], A. C. CUNHA[3] & M. J. D. NASCIMENTO[4]

[1]Universidade Federal de São Carlos, DMAT, SP, Brazil, vini.tavares.azevedo@gmail.com
[2] Universidade de São Paulo, ICMC, SP, Brazil, ebonotto@icmc.usp.br
[3]Universidade Federal da Bahia, IME, BA, Brazil, cunha.arthurc@gmail.com
[4]Universidade Federal de São Carlos, DMAT, SP, Brazil, marcelojdn@ufscar.br

Abstract

We consider a nonautonomous semilinear evolution problem that models some sort of propagation problem in nonlinear elastic rods and nonlinear ion-acoustic waves. We investigate the existence and stability of a family of pullback exponential attractors for our problem under suitable growth and dissipativeness conditions. Moreover, we also prove the upper and lower semicontinuity of this family of pullback exponential attractors at time zero. As a particular case, we obtain the existence of the pullback attractor in an appropriate space, we prove its upper semicontinuity and, lastly, we obtain a regularity result of this pullback attractor.

1 Introduction

In this talk we are interested in the pullback dynamics (and so the asymptotic behavior in the pullback sense) for the family of nonautonomous semilinear evolution problem of second order given by

$$\begin{cases} u_{tt} - \Delta u - \eta_\epsilon(t)\Delta u_t - \Delta u_{tt} = f(u),\ t > s,\ x \in \Omega, \\ u = 0,\ t \geq s,\ x \in \partial\Omega, \\ u(s,x) = u_0(x),\ \ u_t(s,x) = v_0(x),\ x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 3$, $\epsilon \in [0,1]$ is a parameter, $\eta_\epsilon \colon \mathbb{R} \longrightarrow (0,\infty)$ is a continuous function satisfying $0 < a_1 \leq \eta_\epsilon(t) \leq a_2 < \infty$ for all $t \in \mathbb{R}$ (and all $\epsilon \in [0,1]$) with $\lim_{\epsilon\to 0} \|\eta_\epsilon - \eta_0\|_{L^\infty(\mathbb{R})} = 0$, and $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz function satisfying suitable growth and dissipativeness conditions.

Let us denote by $\lambda_1 > 0$ the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in Ω. The system (1) is considered in the Hilbert space $H_0^1(\Omega) \times H_0^1(\Omega)$ and, in this space, we prove the local and global well posedness of solutions and the existence and stability of a family of pullback exponential attractors under the following assumptions:

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1,$$

$$|f(s)| \leq c(1 + |s|^\rho), \qquad s \in \mathbb{R},$$

and

$$|f(s_1) - f(s_2)| \leq c|s_1 - s_2|\big(1 + |s_1|^{\rho-1} + |s_2|^{\rho-1}\big), \qquad s_1, s_2 \in \mathbb{R},$$

for some $c > 0$ and some $1 < \rho < \frac{N+2}{N-2}$.

Our model (1) is motivated by an autonomous counterpart that has been considered by several authors in the last years, but with emphasis in [2] and the references therein. This kind of model has significant physical applications, for instance, it arises to represent some sort of propagation problem in nonlinear elastic rods and nonlinear ion-acoustic waves. In addition, when the term Δu_{tt} is dropped, equation (1) becomes the well-known strongly damped wave equation.

2 Main Results

Let us denote $X = L^2(\Omega)$ and for any $\epsilon \in [0,1]$ let $\{S^{(\epsilon)}(t,s) : t \geq s\}$, with $S^{(\epsilon)}(t,s) : X^{\frac{1}{2}} \times X^{\frac{1}{2}} \longrightarrow X^{\frac{1}{2}} \times X^{\frac{1}{2}}$, be the evolution process associated to (1). The main result of this work is given in the following and guarantees the existence and stability of a family of pullback exponential attractors for the problem (1) (see Theorem 4.11 in [1]).

Theorem 2.1. *For any $\theta \in (\lambda, 1)$ and $\epsilon \in [0,1]$, there exists a pullback exponential attractor $\{\mathcal{M}_\theta^\epsilon(t) : t \in \mathbb{R}\} \subset \mathcal{B} \subset X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ for the evolution process $\{S^{(\epsilon)}(t,s) : t \geq s\}$ with fractal dimension uniformly bounded by*

$$\dim_F \left(\mathcal{M}_\theta^\epsilon(t); V\right) \leq \frac{\ln\left(m_V\left(\frac{2}{\theta-\lambda}\right)\right)}{-\ln\theta}, \quad \textit{for all } t \in \mathbb{R},$$

where $V := X^{\frac{1}{2}} \times X^{\frac{1}{2}}$, $W := X^{\frac{1}{2}-\frac{\gamma}{2}} \times X^{\frac{1}{2}-\frac{\gamma}{2}}$ for $\gamma := 1 - \rho\frac{N-2}{N+2}$ and $m_V(R)$ denotes the maximal number of points z_i in the ball $B_V(0,R)$ such that $\kappa\|z_i - z_j\|_W > 1$. Moreover, the map $\epsilon \mapsto \mathcal{M}_\theta^\epsilon$ is stable in the following sense: given $\epsilon_0 \in [0,1]$, if $\epsilon \in [0,1]$ is such that

$$\Gamma(\epsilon,\epsilon_0) := \sup_{u\in\mathcal{B}} \sup_{t\in\mathbb{R}} \sup_{r\in[0,\widetilde{T}]} \left\|S^{(\epsilon)}(r+t,t)u - S^{(\epsilon_0)}(r+t,t)u\right\|_V < 1$$

then

$$\sup_{t\in\mathbb{R}} \left\{\operatorname{dist}_V^{\mathrm{symm}}\left(\mathcal{M}_\theta^\epsilon(t), \mathcal{M}_\theta^{\epsilon_0}(t)\right)\right\} \leq c\Gamma(\epsilon,\epsilon_0)^\zeta,$$

for some $c > 0$ and $0 < \zeta < 1$ which are independent of ϵ.

Moreover, as some of the consequences of Theorem 2.1 we have (see Corollary 4.12 and Theorem 4.15 in [1]):

Theorem 2.2. *For any $\epsilon \in [0,1]$, the evolution process $\{S^{(\epsilon)}(t,s) : t \geq s\}$ admits a pullback attractor $\{\mathcal{A}^\epsilon(t) : t \in \mathbb{R}\}$ in $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ such that for any given $\theta \in (\lambda,1)$ it holds $\mathcal{A}^\epsilon(t) \subset \mathcal{M}_\theta^\epsilon(t) \subset \mathcal{B}$ for all $t \in \mathbb{R}$ with*

$$\dim_F \left(\mathcal{A}^\epsilon(t); V\right) \leq \frac{\ln\left(m_V\left(\frac{2}{\theta-\lambda}\right)\right)}{-\ln\theta}, \quad \textit{for all } t \in \mathbb{R}.$$

Moreover, $\bigcup_{\epsilon\in[0,1]} \bigcup_{s\in\mathbb{R}} \mathcal{A}^\epsilon(s)$ is bounded in $X^{\frac{1}{2}} \times X^{\frac{1}{2}}$ and the family of pullback attractors $\{\mathcal{A}^\epsilon(t) : t \in \mathbb{R}\}_{\epsilon\in[0,1]}$ is upper-semicontinuous at $\epsilon_0 = 0$, that is, for each $t \in \mathbb{R}$ it holds

$$\lim_{\epsilon\to 0} \left[\operatorname{dist}_{X^{\frac{1}{2}}\times X^{\frac{1}{2}}}\left(\mathcal{A}^\epsilon(t), \mathcal{A}^0(t)\right)\right] = 0.$$

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LOCAL WELL-POSEDNESS AND SPATIAL REGULARITY TO A NON-AUTONOMOUS STRONGLY DAMPED PLATE EQUATION WITH A NONLINEAR MEMORY TERM

BRUNO DE ANDRADE

Universidade Federal de Sergipe, bruno@mat.ufs.br

Abstract

Using the theory of evolution process generated by sectorial operators, we ensure sufficient conditions to the well-posedness and spatial regularity of a non-autonomous strongly damped plate equation with a nonlinear memory term involving singular and regular kernels.

1 Introduction

Damped plate equations with a memory term has been de subject of several research papers in the last years. In this work, we consider the following non-autonomous damped plate equation with nonlinear memory effect

$$\begin{cases} u_{tt} + \beta(t)u_t = -\Delta^2 u + \int_0^t a(t-s)\sigma((-\Delta)^\gamma u(s,x))ds + h(u), \text{ in } [0,\infty)\times\Omega, \\ u(t,x) = 0, \text{ on } [0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \text{ in } \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $\gamma \geq 0$. The kernel $a : (0,\infty) \to \mathbb{R}$ verifies

$$|a(t)| \leq kt^\rho, \quad t > 0, \tag{2}$$

with $\rho > -1$. The function $\beta : \mathbb{R} \to \mathbb{R}$ is a ϑ-Hölder continuous function such that

$$\beta_0 \leq \beta(t) \leq \beta_1, \quad \forall\, t \in \mathbb{R},$$

for some $\beta_0, \beta_1 \in (0,\infty)$. Furthermore, $\sigma : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous functions that verify

$$|\sigma(r) - \sigma(s)| \leq C_2(|r|^{p_1-1} + |s|^{p_1-1})|r-s|, \quad \forall r,s \in \mathbb{R}, \tag{3}$$

and

$$|h(r) - h(s)| \leq C_1(|r|^{p_2-1} + |s|^{p_2-1})|r-s|, \quad \forall r,s \in \mathbb{R}, \tag{4}$$

for some $C_i > 0$ and $p_i > 1$, $i = 1, 2$.

Partial integrodifferential equations of type (1) arise in many physical contexts. For example, an autonomous linear version of (1) was deduced by Ferreira et al. in the situation of a material that has viscoelastic properties modeled by a Maxwell-Wiechert model. Similarly, in the one dimensional case, Nohel considers a version of (1) (without damxping) as a mathematical model for the motion of nonlinear viscoelastic rods. Naturally, several others researchers consider problems of type (1). It is important to note that almost all these works suppose that the kernel a is a bounded function (regular kernels). Indeed, the typical hypothesis is to consider this function as a linear combination of decaying exponentials with positive coefficients. Using the theory of evolution process generated by sectorial operators, we ensure sufficient conditions to the well-posedness and spatial regularity of a non-autonomous strongly damped plate equation with a nonlinear memory term involving singular and regular kernels.

2 Main Results

By using an abstract framework, problem (1) can be rewritten ia suitable Banch space as the following integro-differential equation

$$\begin{cases} w' = \mathcal{A}(t)w + \int_0^t a(t-s)g(w(s))ds + f(w(t)), \ t \geq 0, \\ w(0) = w_0, \end{cases} \tag{5}$$

where $w = \begin{pmatrix} u \\ u' \end{pmatrix}$, $\mathcal{A}(t)$ is a suitable sectorial operator, and the functions g and f are given by

$$g(\psi) = \begin{pmatrix} 0 \\ \sigma((-\Delta)^\gamma \psi_2) \end{pmatrix}$$

andx

$$f(\psi) = \begin{pmatrix} 0 \\ h(\psi_2) \end{pmatrix},$$

for all $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in X^1 = H_0^1(\Omega) \times L^2(\Omega)$.

The main result of this work ensures:

Theorem 2.1. *Let* $\max\{1-\frac{N}{2}, 0\} < \alpha < 1 < p_1, p_2 \leq 1 + \frac{2}{N}(1-\alpha)$ *and* $0 \leq \gamma \leq \frac{1-\alpha}{2p_1} + \frac{N(1-p_1)}{4p_1}$*. Given* $v_0 \in X^1$*, we can consider* $r > 0$ *and* $\tau > 0$ *such that for any* $w_0 \in B_{X^1}(v_0, r)$ *there exists a unique mild solution* $w \in C([0,\tau]; X^1)$ *to problem* (5)*. Furthermore, for all* $0 \leq \theta < \alpha$ *it follows that*

$$w \in C((0,\tau]; X^{1+\theta})$$

and if $\theta > 0$ *then*

$$\lim_{t\to 0^+} t^\theta \|w(t, w_0)\|_{X^{1+\theta}} = 0.$$

Moreover, if $w_0, w_1 \in B_{X^1}(v_0, r)$*, then there exists a constant* $C > 0$ *such that*

$$t^\theta \|w(t, w_0) - w(t, w_1)\|_{X^{1+\theta}} \leq C\|w_0 - w_1\|_{X^1}, \ \forall t \in [0,\tau], \ 0 \leq \theta \leq \theta_0 < \alpha.$$

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CARLEMAN INEQUALITY FOR A CLASS OF SUPER STRONG DEGENERATE PARABOLIC OPERATORS AND APPLICATIONS

B. S. V. ARAÚJO[1], R. DEMARQUE[2], & L. VIANA[3]

[1]Universidade Federal de Campina Grande, UAMat, PB, Brazil, bsergio@mat.ufcg.edu.br
[2]Universidade Federal Fluminense, Departamento de Ciências da Natureza, RJ, Brazil, reginaldo@id.uff.br
[3]Universidade Federal Fluminense, Departamento de Análise, RJ, Brazil, luizviana@id.uff.br

Abstract

In this paper, we present a new Carleman estimate for the adjoint equations associated to a class of super strong degenerate parabolic linear problems. Our approach considers a standard geometric imposition on the control domain, which can not be removed in general. Additionally, we also apply the aforementioned main inequality in order to investigate the null controllability of two nonlinear parabolic systems. The first application is concerned a global null controllability result obtained for some semilinear equations, relying on a fixed point argument. In the second one, a local null controllability for some equations with nonlocal terms is also achieved, by using an inverse function theorem.

1 Introduction

In this work we derive a new Carleman estimate for the linear super strong degenerate problem

$$\begin{cases} u_t - (x^\alpha u_x)_x + x^{\alpha/2} b_1(x,t) u_x + b_0(x,t) u = f 1_\omega & \text{in } Q, \\ u(1,t) = 0 \text{ and } (x^\alpha u_x)(0,t) = 0 & \text{in } (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1), \end{cases} \tag{1}$$

where $Q = (0,1) \times (0,T)$, $\omega \subset (0,1)$ is a non-empty open interval and 1_ω is its associated characteristic function, and $\alpha \geq 2$. Also, we take $b_0 \in L^\infty(Q)$, $h \in L^2(\omega \times (0,T))$, $u_0 \in L^2(0,1)$, and $b_1 \in L^\infty(Q)$ satisfying

$$(x^{\alpha/2} b_1(x,t))_x \in L^\infty(Q). \tag{2}$$

We also consider a geometrical condition on the control domain

$$\exists d > 0;\ (0,d) \subset \omega. \tag{3}$$

We say that (1) is *null controllable* if, for any $u_0 \in L^2(0,1)$, there exists a control $h \in L^2(\omega \times (0,T))$ such that the solution u of (1) satisfies

$$u(\cdot,T) = 0.$$

The null controllability of (1) is well understood for $\alpha \in (0,2)$, even without the geometric condition (3) being imposed, see [1, 3] and references therein. Following the terminology adopted in these works, we say that (1) is *weakly degenerate* if $\alpha \in (0,1)$ and *strongly degenerate* if $\alpha \in (1,2)$. Despite there are many works for the case $\alpha \in (0,2)$, little has been done for the *super strong degenerate case*, i.e. when $\alpha \geq 2$, although this is a very relevant case of the degenerate problem. Indeed, when $\alpha = 2$, the Black-Scholes equation can be obtained from (1) and this equation has a key role in several financial applications.

2 Main Results

First of all, let us consider the adjoint system associated to (1):

$$\begin{cases} v_t + (x^\alpha v_x)_x + (x^{\alpha/2} b_1 v)_x - b_0(x,t)v = h & \text{in } Q, \\ v(1,t) = 0 \ \text{ and } \ (x^\alpha v_x)(0,t) = 0 & \text{in } (0,T), \\ v(x,T) = v_T(x) & \text{in } (0,1), \end{cases} \tag{4}$$

where $h \in L^2(Q)$ and $v_T \in L^2(0,1)$.

Now, for $\lambda > 0$, let us introduce some weight functions given by

$$\theta(t) := \frac{1}{(t(T-t))^4}, \ \eta(x) := -x^2/2, \ \xi(x,t) = \theta(t)e^{\lambda(2|\eta|_\infty + \eta(x))} \ \text{ and } \ \sigma(x,t) := \theta(t)e^{4\lambda|\eta|_\infty} - \xi(x,t). \tag{5}$$

The geometrical assumption (3) and the weight function η are the key points that allow us to build the following Carleman estimate:

Theorem 2.1. *Assume* (2) *and* (3). *There exists positive constants* C, s_0 *and* λ_0, *depending only on* ω, $\|b_0\|_\infty$, T, d *and* α *such that, for any* $s \geq s_0$, *any* $\lambda \geq \lambda_0$ *and any solution* v *to (4), one has:*

$$\iint_Q e^{-2s\sigma}\left[s^{-1}\lambda^{-1}\xi^{-1}(|v_t|^2 + |(x^\alpha v_x)_x|^2) + s\lambda^2\xi x^\alpha |v_x|^2 + s^3\lambda^4\xi^3|v|^2\right] dx\,dt$$
$$\leq C\left[\|e^{-s\sigma}h\|^2 + s^3\lambda^4 \iint_{\omega_T} e^{-2s\sigma}\xi^3|v|^2\,dx\,dt\right], \tag{6}$$

where $\omega_T := \omega \times (0,T)$.

As a consequence of Theorem 2.1 we have the following null controllability result:

Theorem 2.2. *Assume* (2) *and* (3). *Then the system* (1) *is null controllable.*

Following the ideas of [3] we can extend the Theorem 2.1 to semilinear problems. Sistems with nonlocal operators can also be treated using the framework present in [2], where the authors obtain local null controllability results.

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ORBITAL STABILITY OF KINK-TYPE WAVES FOR A DEFOCUSING NONLINEAR SCHRÖDINGER MODEL

C. M. MEDEIROS[1] & A. J. CORCHO[2]

[1]Universidade Federal do Rio de Janeiro, UFRJ, RJ, Brazil, carlosjuarezmartins@gmail.com
[2]Universidad de Córdoba - UCO, España, a.corcho@uco.es

Abstract

In this work we present a detailed proof of orbital stability for a family of kink standing waves solutions for a non-linear Schrödinger equation on defocusing regime, which models several phenomena in Mathematical Physics. The kink waves often called dark solitons in the literature are, in particular, non localized solutions of the model non-vanishing at infinity. The proof we will give is based on the results presented by P. Zhidkov in [1]. The theory is developed taking a specific nonlinearity in order to facilitate reading. However, the prove presented in the particular case can be extended, with minor changes, to a more general context of nonlinearity for the model.

1 Introduction

We are interested in studying the orbital stability of certain types of waves that are solutions to the initial value problem (IVP) of the Nonlinear Schrödinger Equation (NLS):

$$iu_t + u_{xx} + f(|u|^2)u = 0,\ x \in \mathbb{R},\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{2}$$

where $u = u(x,t) \in \mathbb{C}$ with $(x,t) \in \mathbb{R}^2$, $f(\cdot)$ being a sufficiently regular real function.

For our analysis we will assume u to be of the standing wave type, that is, $u(x,t) = e^{i\omega t}\phi(x)$. Substituting this function into (1), we arrive at the following ODE $\phi'' - \omega\phi + f(\phi^2)\phi = 0$. The main theorem also assumes that ϕ is a kink with non-negative limits, that is, a solutions that do not vanish at infinity and have limits greater than or equal to zero. Setting $\lim_{x\to\pm\infty}\phi(x) =: \phi_\pm$ it is demonstrated that the necessary and sufficient conditions for the existence of these kinks for NLS are:

1. $\phi_-, \phi_+ \geq 0$, $\phi_- \neq \phi_+$;
2. $-\omega + f(\phi_\pm^2) + 2\phi_\pm^2 f'(\phi_\pm^2) < 0$;
3. $-\omega\phi_\pm + f(\phi_\pm^2)\phi_\pm = 0$;
4. $-\frac{\omega}{2}\phi_-^2 + U(\phi_-^2) = \frac{\omega}{2}\phi_+^2 + U(\phi_+^2)$;
5. $-\frac{\omega}{2}s^2 + U(s^2) < \frac{\omega}{2}\phi_-^2 + U(\phi_-^2)$ for all $s \in (\phi_-, \phi_+)$;

a classic model that satisfies these properties is the cubic-quintic model, that is, with $f_1(s) = s - s^2$.

For the proof of the theorem to be more instructive for readers who do not yet have an advanced understanding of spectral theory and Sobolev embeddings, we will fix

$$f(s) := f_\omega(s) = (\sqrt{s} - 1)(2 - \sqrt{s}) + \omega,\ \omega \in \mathbb{R}, \tag{3}$$

which, substituting into (2), we can obtain the kink-type solution $\phi(x) = 1 + \tanh\left(\frac{x}{\sqrt{2}}\right)$.

Then $u_\omega(x,t) = e^{i\omega t}\phi(x)$ is a solution of (1) associated with $f = f_\omega$ with initial value $u_\omega(x,0) = \phi(x)$.

A suitable functional space to study the dynamics of kink-type solutions for (1) is the so-called Zhidkov spaces, defined as follows:

Definition 1.1. *Given $k \in \mathbb{N}$ the space $X^k(\mathbb{R})$ is the closure of the space*

$$\left\{ v \in L^\infty(\mathbb{R}) \cap C^k(\mathbb{R}),\ v \text{ isÂ absolutelyÂ continuousÂ and } v' \in H^{k-1}(\mathbb{R}) \right\}$$

with respect to the norm $\|v\|_{X^k} = \|v\|_{L^\infty} + \sum_{i=1}^{k-1} \left\| \frac{d^i v}{dx^i} \right\|_{L^2}$, where $H^{k-1}(\mathbb{R})$ denotes the classical Sobolev space of index $k-1$, based on the space $L^2(\mathbb{R})$.

Now, note that $f_\omega(|z|^2)z$ is not holomorphic, so the classical result of well-posedness for the IVP (1)-(2) presented in [1] does not apply. To solve this problema, we use perturbations of u_ω by functions $\psi(\cdot, t) \in H^1(\mathbb{R})$ and the well-posedness theory developed in [2].

2 Main Results

Now let

$$R(\tau) = \|u(\cdot - \tau, t) - \phi(\cdot)\|_{H^1} \tag{4}$$

and τ_0 such that $R(\tau_0)$ is the minimum value of $R(\tau)$. Finally, we define

Definition 2.1. *With τ_0 defined above, we define*

$$v(x,t) := u(x - \tau_0, t), \tag{5}$$

$$z(x,t) := |v(x,t)| - \phi(x). \tag{6}$$

Now, let $u \in X^1$ be a solution of (1) such that $u(\cdot, 0) - \phi(x) \in H^1$. Additionally, let z be such that $\|z(\cdot, t)\|_{H^1}$ is small enough so that $0 < \phi(x) + z(x,t) < c < \infty$ for all x, Then we can use the following complex representation of v:

$$v(x,t) = (\phi(x) + z(x,t))e^{i[\omega t + \tilde{\omega}(x,t)}, \tag{7}$$

where $\tilde{\omega}(x,t)$ is a real absolutely continuous, bounded, and periodic function with period $2\pi m, m \in \mathbb{Z}$. Obviously $v \in X^1$ and

$$v_x(x,t) = [\phi'(x) + z_x(x,t) + i(\phi(x) + z(x,t))\tilde{\omega}(x,t)]e^{[i[\omega t + \tilde{\omega}(x,t)}. \tag{8}$$

Now, we use the above representations and the following energy functional

$$E(u) = \int_{\mathbb{R}} \left\{ \frac{1}{2}|u_x(x)|^2 - U(|u(x)|^2) + \frac{\omega}{2}|u(x)|^2 + D \right\} dx \tag{9}$$

where $U(s) = \frac{1}{2}\int_0^s f(r)dr$ and $D = -\frac{\omega}{2}\phi_-^2 + U(\phi_-^2)$, to prove the main theorem:

Theorem 2.1. *Let $f \in C^2(\mathbb{R}^+)$ and suppose that conditions 1-5 are satisfied. Then the kink $u_\omega(x,t) = e^{i\omega t}\phi(x)$ of the NLS (1) is stable in the following sense:*

For every $\epsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in X^1$, $|u_0(\cdot)| - \phi(\cdot) \in H^1$, $\|z(\cdot, 0)\|_{H^1} < \delta$ and $\|\tilde{\omega}_x(\cdot, 0)\|_{L^2} < \delta$, then the solution $u(x,t) \in X^1$ of the problem (1)-(2) is global. Moreover,

$$|u(\cdot, t)| - \phi(\cdot) \in H^1, \quad \|z(\cdot, t)\|_{H^1} < \epsilon \quad \textit{and} \quad \|\tilde{\omega}_x(\cdot, t)\|_{L^2} < \epsilon, \ \textit{for all } t > 0. \tag{10}$$

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DECAY RATES FOR A WEAKLY DAMPED COUPLED WAVE EQUATIONS IN $\mathbb{R}^N$

G. DE MARTINI & C. R. DA LUZ

[1]Federal University of Santa Catarina, DM, SC, Brazil, guilhermedemartini@live.com, cleverson.luz@ufsc.br

Abstract

An important research matter is the search for minimal damping in such a way that the total energy associated to the model decays uniformly to zero as the time tends to infinity. We say that a system is weakly dissipative when as least one of the equations that make up the system has no dissipative term. In this work, we consider the a coupled wave equations in $\mathbb{R}^n$ without a damping term in the second equation. We studied the asymptotic behavior of the total energy associated to the linear system (1)-(2). The method developed in this work is based on the energy of the Fourier space, the monotonicity of the local and total energy in the Fourier space and the property of integrability of certain singularities around the origin. The approach developed in this work can be applied in the study of the asymptotic behavior of other linear, weakly dissipative, systems in $\mathbb{R}^n$.

1 Introduction

In our discussion we consider the following evolution system:

$$\begin{aligned} u_{tt}(t,x) - \Delta u(t,x) + \beta u_t(t,x) + \lambda u(t,x) - \kappa v(t,x) = 0, \\ v_{tt}(t,x) - \Delta v(t,x) + \mu v(t,x) - \kappa u(t,x) = 0, \end{aligned} \tag{1}$$

for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$, where β, λ, μ, κ are constants. The initial data are given by

$$\begin{aligned} u(0,x) = u_0(x) \quad \text{and} \quad u_t(0,x) = u_1(x), \\ v(0,x) = v_0(x) \quad \text{and} \quad v_t(0,x) = v_1(x), \end{aligned} \tag{2}$$

for all $x \in \mathbb{R}^n$. The above model can be used to describe the evolution of a system consisting of two elastic membranes subject to an elastic force that attracts one membrane to the other.

In order to obtain decay rates for the total energy, we work with the system (1) in the Fourier space. For each fixed $t > 0$, we apply the Fourier transform with respect to the space variable x:

$$\hat{u}_{tt}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) + \beta \hat{u}_t(t,\xi) + \lambda \hat{u}(t,\xi) - \kappa \hat{v}(t,\xi) = 0, \tag{3}$$

$$\hat{v}_{tt}(t,\xi) + |\xi|^2 \hat{v}(t,\xi) + \mu \hat{v}(t,\xi) - \kappa \hat{u}(t,\xi) = 0, \tag{4}$$

for all $\xi \in \mathbb{R}^n$ and $t > 0$. The initial data in the Fourier space are given by

$$\begin{aligned} \hat{u}(0,\xi) = \hat{u}_0(\xi) \quad \text{and} \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \\ \hat{v}(0,\xi) = \hat{v}_0(\xi) \quad \text{and} \quad \hat{v}_t(0,\xi) = \hat{v}_1(\xi), \end{aligned}$$

for all $\xi \in \mathbb{R}^n$.

We multiply both sides of (3) by $\overline{\hat{u}}_t$ and multiply (2) by $\overline{\hat{v}}_t$. Then, taking the real part and adding the resulting equations, we obtain the following identity

$$\frac{d}{dt}(\mathcal{E}(t,\xi)) + \beta |\hat{u}_t(t,\xi)|^2 = 0, \quad \xi \in \mathbb{R}^n \text{ and } t > 0,$$

where

$$\mathcal{E}(t,\xi)=\frac{1}{2}\left\{|\hat{u}_t(t,\xi)|^2+\left(|\xi|^2+\lambda\right)|\hat{u}(t,\xi)|^2+|\hat{v}_t(t,\xi)|^2+\left(|\xi|^2+\lambda\right)|\hat{v}(t,\xi)|^2-\kappa Re\left[\hat{u}(t,\xi)\,\overline{\hat{v}}(t,\xi)\right]\right\}.$$

We define the total energy of the system, in Fourier space, as

$$E(t):=\frac{1}{2}\int_{\mathbb{R}^n}\Big(|\hat{u}_t(t,\xi)|^2+\left(|\xi|^2+\lambda\right)|\hat{u}(t,\xi)|^2+|\hat{v}_t(t,\xi)|^2+\left(|\xi|^2+\lambda\right)|\hat{v}(t,\xi)|^2-\kappa Re\left[\hat{u}(t,\xi)\,\overline{\hat{v}}(t,\xi)\right]\Big)\,d\xi,$$

for any $t\geq 0$.

2 Main Results

The main result of this work is the following theorem on the asymptotic behavior of solutions (with decay rates) for the system (1)-(2).

Theorem 2.1. *Let $n\geq 1$ and β, λ, μ,κ constants satisfying suitable conditions. If $[u_0,u_1,v_0,v_1]\in H^{1+\frac{1}{\alpha}}(\mathbb{R}^n)\times H^{\frac{1}{\alpha}}(\mathbb{R}^n)\times H^{1+\frac{1}{\alpha}}(\mathbb{R}^n)\times H^{\frac{1}{\alpha}}(\mathbb{R}^n)$, then there exists a constant $C_\alpha>0$ depending on α, such that the total energy of the system (1)-(2), in Fourier space, satisfies*

$$E(t)\leq C_\alpha\big\{\|u_0\|^2_{H^{1+\frac{1}{\alpha}}}+\|u_1\|^2_{H^{\frac{1}{\alpha}}}+\|v_0\|^2_{H^{1+\frac{1}{\alpha}}}+\|v_1\|^2_{H^{\frac{1}{\alpha}}}\big\}\,t^{-1/\alpha},$$

for all $t\geq T_0$ where T_0 is a constant depending on the initial data.

As an immediate consequence of Theorem 2.1 the following result follows:

Corollary 2.1. *Considering the same assumptions of Theorem 2.1 then there exists a constant $C_\alpha>0$ depending on α, such that the unique solution $(u(t,x),v(t,x))$ of the system (1)-(2) satisfy:*

$$\|u(t)\|^2_{H^1}+\|u_t(t)\|^2_{L^2}+\|v(t)\|^2_{H^1}+\|v_t(t)\|^2_{L^2}\leq C_\alpha\big\{\|u_0\|^2_{H^{1+\frac{1}{\alpha}}}+\|u_1\|^2_{H^{\frac{1}{\alpha}}}+\|v_0\|^2_{H^{1+\frac{1}{\alpha}}}+\|v_1\|^2_{H^{\frac{1}{\alpha}}}\big\}\,t^{-1/\alpha},$$

for all $t\geq T_0$ where T_0 is a constant depending on the initial data.

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MILD SOLUTIONS IN ONE THE FRACTIONAL NAVIER-STOKES-CORIOLIS EQUATION IN MORREY SPACES

B. DE ANDRADE[1], C. B. AMORIM[2], & E. MATEUS[2]

[1]Universidade Federal de Sergipe, DMA, SE, Brazil, bruno@mat.ufs.br

[2]Universidade Federal de Sergipe, DMAI, SE, Brazil, charles@academico.ufs.br, edermateus@mat.ufs.br

Abstract

We establish the local well-posedness of the fractional Boussinesq-Coriolis system on Morrey Space. For this,we used of the Mittag-Leffler families $\{E_\alpha(t)\}_{t\geq 0}$ and $\{E_{\alpha,\alpha}(t)\}_{t\geq 0}$ for equation (1.1), establish their behavior on the scale of Morrey spaces and obtain asymptotic estimates for such families.

1 Introduction

In this work we are interested in studying the initial value problem for the fractional equations

$$\begin{cases} \partial_t^\alpha u + \nu(-\Delta)^\beta u + \Omega e_3 \times u + (u\cdot\nabla)u + \nabla p = g\omega e_3, & (t,x)\ \mathbb{R}\times\mathbb{R}^3, \\ \partial_t^\alpha \theta + \mu(-\Delta)^\beta \theta + (u\cdot\nabla)\theta = -\mathcal{N} u_3, & (t,x)\ \mathbb{R}^+\times\mathbb{R}^3, \\ \nabla\cdot u = 0, & \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1}$$

where ∂_t^α is Caputo's fractional derivative of order $\alpha\in(0,1]$ and $\frac{1}{2}\leq\beta<\frac{5}{2}$. When $\alpha=1$, these equations represent the 3D fractional Boussinesq-Coriolis equations with stratification. In this context μ is the viscosity, $p=p(x,t)$ is the pressure of the fluid and θ is a scalar function that represents the buoyancy density in the fluid (in the case of the ocean this function depends temperature and salinity, and in the case of the atmosphere it depends on temperature). The initial data $u_0=u_0(x)=(u_0^1(x),u_0^2(x),u_0^3(x))$ denotes the initial velocity field satisfying the compatibility condition $\nabla\cdot u=0$.The constants ν, μ and g are related to viscosity, diffusivity and gravity, respectively. The constant $\Omega\neq 0$ represents the speed of rotation around the vertical unit vector $e_3=(0,0,1)$ and is called the Coriolis parameter. The stratification parameter $\mathcal{N}$ is a non-negative constant that represents the frequency of the Brunt-Väisälä wave. The proportion $P=\frac{\mu}{\nu}$ is known as the Prandtl number and $B=\frac{\Omega}{\mathcal{N}}$ is essentially the Burger number of geophysics.

The main objectives of this work are to guarantee the existence of local mild solutions for equation (1.1) on the scale Morrey spaces finds solutions based on estimates of the functions of of the Mittag-Leffler families $\{E_\alpha(t)\}_{t\geq 0}$ and $\{E_{\alpha,\alpha}(t)\}_{t\geq 0}$ for equation (1.1). These families have behavior on the scale of Morrey spaces and asymptotic estimates .

Morrey spaces are defined as follows. Let $B_d(x_0)$ be the open ball in $\mathbb{R}^n$ centered at x_0 and with radius $d>0$. For $1\leq q<\infty$ and $0\leq\mu<n$, the homogeneous Morrey space $\mathcal{M}_{q,\mu}=\mathcal{M}_{q,\mu}(\mathbb{R}^n)$ is the space of all $f\in L^q_{\text{loc}}$ such that

$$\|f\|_{q,\mu}=\sup_{x_0\in\mathbb{R}^n,d>0} d^{-\frac{\mu}{q}}\|f\|_{L^q(B_d(x_0))}<\infty.$$

In the case $q=1,\mathcal{M}_{1,\mu}$ is a subspace of Radon measures and the L^1-norm in (2.1) should be understood as the total variation of the measure f on $B_d(x_0)$. The space $\mathcal{M}_{q,\mu}$ endowed with $\|\cdot\|_{q,\mu}$ is a Banach space. For more details, we refer the reader to [2] and their references.

2 Main Results

Considering $N = \mathcal{N}\sqrt{g}, v = \left(v^1, v^2, v^3, v^4\right) = \left(u^1, u^2, u^3, \sqrt{g}\theta/\mathcal{N}\right)$, $v_0 = \left(v_0^1, v_0^2, v_0^3, v_0^4\right) = \left(u_0^1, u_0^2, u_0^3, \sqrt{g}\theta_0/\mathcal{N}\right)$, and $\tilde{\nabla} = (\partial_1, \partial_2, \partial_3, 0)$, we can convert the above system as

$$\begin{cases} \partial_t^\alpha v + \mathcal{A}v + \mathcal{B}v + \tilde{\nabla}p = -(v\cdot\tilde{\nabla})v, \text{ in } \mathbb{R}^3\times(0,\infty) \\ \tilde{\nabla}\cdot v = 0, \text{ in } \mathbb{R}^3\times(0,\infty) \\ v(x,0) = v_0(x), \text{ in } \mathbb{R}^3 \end{cases}$$

where

$$\mathcal{A} = \begin{pmatrix} \nu(-\Delta)^\beta & 0 & 0 & 0 \\ 0 & \nu(-\Delta)^\beta & 0 & 0 \\ 0 & 0 & \nu(-\Delta)^\beta & 0 \\ 0 & 0 & 0 & k(-\Delta)^\beta \end{pmatrix} \text{ and } \mathcal{B} = \begin{pmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -N \\ 0 & 0 & N & 0 \end{pmatrix}.$$

A mild solution of this problem is a function that verifies the integral equation

$$v(t) = E_\alpha(t)v_0 - \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(t-s)\mathbb{P}(v\cdot\tilde{\nabla})v ds, \quad t\geq 0, \tag{2}$$

where $\mathbb{P}$ is the Leray Projector.

Theorem 2.1. *Let $\alpha\in(0,1]$, $I=(0,+\infty)$, $2\leq q\leq\infty$, $\max\{0,3-q\}<\mu<3$. For each par $(\Omega,\mathcal{N})\in(\mathbb{R}-\{0\})^2$, consider $L=\max\left\{2,\frac{|\Omega|}{\mathcal{N}\sqrt{g}},\frac{\mathcal{N}\sqrt{g}}{|\Omega|}\right\}$. Then, there exist a local mild solution de (1.1) in $BC((0,T),\mathcal{M}_{q_1,\mu})$*

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EXISTENCE OF STATIONARY VORTEX PATCHES FOR THE GSQG IN BOUNDED DOMAINS

V. ANGULO-CASTILLO[1], E. CUBA[2], & LUCAS C. F. FERREIRA[3]

[1]Department of Mathematics, National University of Colombia, Arauca, Colombia, vlcastillo@unal.edu.co
[2]Department of Mathematics, State University of Campinas, Campinas, São Paulo, Brazil, ecubah@ime.unicamp.br
[3]Department of Mathematics, State University of Campinas, Campinas, São Paulo, Brazil, lcff@ime.unicamp.br

Abstract

We show the existence of time-periodic vortex patches for the generalized surface quasi-geostrophic equation within a bounded domain. This construction is carried out for values of γ in the range of $(1, 2)$. The resulting vortex patches possess a fixed vorticity and total flux, and they are located in the neighborhood of critical points that are non-degenerate for the Kirchhoff-Routh equation. The proof is accomplished through a combination of analyzing the linearization of the contour dynamics equation and employing the implicit function theorem as well as carefully selected function spaces.

1 Introduction

In this talk, we study the generalized surface quasi-geostrophic (gSQG) equation, which are defined within a bounded domain defined by

$$\begin{cases} \partial_t\omega + \mathbf{v}\cdot\nabla\omega = 0 & \text{in } \Omega\times(0,T), \\ \mathbf{v} = \nabla^{\perp}(-\Delta)^{-1+\frac{\gamma}{2}}\omega & \text{in } \Omega\times(0,T), \\ \omega\big|_{t=0} = \omega_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in two-dimensional space, and we consider a parameter γ satisfying the condition $0 \le \gamma < 2$. The variable $\omega(\boldsymbol{x}, t)$, defined for $\boldsymbol{x}$ within Ω and t in the interval $(0, T)$, represents an active scalar being advected by a velocity field $\mathbf{v}(\boldsymbol{x}, t)$. This velocity field is generated by ω, and $\nabla^{\perp} = (\partial_2, -\partial_1)$. The operator denoted by $(-\Delta)^{-1+\frac{\gamma}{2}}$ is defined as

$$(-\Delta)^{-1+\frac{\gamma}{2}}\omega(\boldsymbol{x}) = \int_{\Omega} K_{\gamma}(\boldsymbol{x},\boldsymbol{y})\omega(\boldsymbol{y}), d\boldsymbol{y},$$

where the term $K_{\gamma}(\boldsymbol{x}, \boldsymbol{y})$ represents the Green function associated with the fractional Laplacian in bounded domains with smooth boundaries. It is defined for each pair of points $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, where $\boldsymbol{x} \neq \boldsymbol{y}$, as follows:

$$K_{\gamma}(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} -\frac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{y}| + K_0^0(\boldsymbol{x},\boldsymbol{y}), & \gamma = 0, \\ \frac{C_{\gamma}}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma}} + K_{\gamma}^0(\boldsymbol{x},\boldsymbol{y}), & \gamma \in (0,2), \end{cases}$$

with $\Gamma(\cdot)$ being the Euler gamma function and $C_{\gamma} = \frac{2^{\gamma-1}\Gamma(\frac{\gamma}{2})}{\Gamma(1-\frac{\gamma}{2})}$. Additionally, K_{γ}^0 belongs to the class of infinitely differentiable functions $C^{\infty}(\Omega\times\Omega)$, as discussed in [1, Lemma 2.3].

Hmidi *et al.* in [1] proved for the gSQG equation (1) the existence of the V-states with $\gamma \in (0, 1)$ in the unit disc, and then, Cao *et al.* in [1] demonstrated the existence of stationary vortex patches. These patches maintained both fixed vorticity and a consistent total flux for each patch. They achieved this within the context of the SQG equation, considering a general bounded domain.

For a collection of m real numbers $\kappa_1, \kappa_2, \ldots, \kappa_m$, we establish the Kirchhoff-Routh function on Ω^m in the following manner

$$\mathcal{W}_m(x_1, x_2, .., x_m) = -\sum_{i \neq j}^{m} \kappa_i \kappa_j K_\gamma^1(x_i, x_j) + \sum_{i=1}^{m} \kappa_i^2 K_\gamma^0(x_i, x_i), \tag{2}$$

where Ω^m is the set of vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ such that each x_i belongs to the set Ω for $i = 1, 2, \ldots, m$ and $K_\gamma^1(x, y) = \frac{C_\gamma}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}$.

2 Main Results

Theorem 2.1. *Consider a bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary and m given positive values κ_i $(i = 1, \ldots, m)$. Assume that $\boldsymbol{x}_0 = (x_{0,1}, \ldots, x_{0,m}) \in \Omega^m$, with $x_{0,i} \neq x_{0,j}$ for $i \neq j$, is an isolated critical point of $\mathcal{W}_m$ as defined in (3) and satisfies the nondegeneracy condition: deg$(\nabla \mathcal{W}_m, \boldsymbol{x}_0) \neq 0$. Under these conditions, there exists $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon < \varepsilon_0$, a stationary vortex patch solution ω_ε can be constructed, which exhibits the following characteristics:*

(i) $\omega_\varepsilon = \sum_{i=1}^m \frac{1}{\varepsilon^2} \chi_{\Gamma_i}$ *within specific domains* $\Gamma_i \subset \Omega, i = 1, \ldots, m$.

(ii) The boundaries $\partial\Gamma_i$ *for* $i = 1, \ldots, m$ *can be defined using the subsequent parameterization*

$$\partial\Gamma_i = \left\{ x_{\varepsilon,i} + \varepsilon \left(\sqrt{\frac{\kappa_i}{\pi}} + o(1) \right) (\cos\beta, \sin\beta) \mid \beta \in [0, 2\pi) \right\},$$

where $x_{\varepsilon,i} = x_{0,i} + o(1)$ *as* $\varepsilon \to 0$.

(iii) The total flux for each patch remains fixed as

$$\frac{1}{\varepsilon^2} |\Gamma_i| = \kappa_i, \quad \forall i = 1, \ldots, m.$$

(iv) As $\varepsilon \to 0^+$, *one has the following convergence in the sense of measures*

$$\omega_\varepsilon \to \sum_{i=1}^{n} \delta(x - x_{0,i}) \text{ weakly,}$$

where $\delta(x - x_{0,i})$ *represents the Dirac delta function concentrates at the point* $x_{0,i}$.

(v) The interior of each domain Γ_i *is convex, for every* $i = 1, \ldots, m$.

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NONLINEAR DIFUSSION EQUATION INVOLVING P(B(U))-LAPLACIAN-LIKE OPERATOR

EUGENIO CABANILLAS L.

Universidad Nacional Mayor de San Marcos, FCM, Lima, Perú, cleugenio@yahoo.com

Abstract

The aim of this paper is to study the existence of weak solutions for a nonlinear difussion equation involving p(u)-Laplacian-like operators. We establish our results by using the time-discretization method and energy methods combined with a singular perturbation technique and Schauder's fixed-point theorem.

1 Introduction

This article is concerned with the existence of weak solutions for the following local $p(u)$-Laplacian problem

$$
\begin{aligned}
u_t - \operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u + \frac{|\nabla u|^{2p(b(u))-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(b(u))}}} \right) + g(u) = f \qquad & \text{in } \Omega, \\
u = 0 \qquad\qquad & \text{on } \partial\Omega,
\end{aligned} \tag{1}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $N \geq 2$, $p \in C(\overline{\Omega})$ for any $x \in \overline{\Omega}$; f, g are given functions. Nonlinear boundary value problems with nonlinearities and nonstandard $p(x)$-growth conditions arise from a variety of physical phenomena such as non-Newtonian fluids, reaction-diffusion problems, petroleum extraction, flow through porous media, etc. Thus, the study of such problems and their generalizations have attracted numerous attention in recent years. Our main interest in this research is when the function p is composed with another function that depends on the unknown solution u (see [1, 5]). Thus, the problem becomes nonlocal and particularly interesting. Parabolic equations involving the $p(u)$-Laplacian have been proposed in the study of image restoration (see [2, 6]) as well as in some model of electrorheological fluids (see [1]). As far as the parabolic type p(u)-Laplacian equations are concerned, only very few papers have appeared (see [2]).

2 Notations and Main Results

Let $p : \mathbb{R} \to [1, +\infty[$ be the nonlinear exponent function such that

$$
p \quad \text{is a Lipschitz-continuous function}\ , \qquad \text{and } 1 < \alpha < p(x) \leq \beta < \infty \quad \text{for a.e. } x \in \Omega. \tag{2}
$$

We consider a mapping $b : W_0^{1,\alpha}(\Omega) \to \mathbb{R}$ such that

$$
b \quad \text{is continuous and bounded}\ . \tag{3}
$$

Here, we note that $p(b(u))$ is here a real number and not a function, then the Sobolev spaces involved in this work are the classical ones. We will consider the well-known Sobolev space $W_0^{1,p}(\Omega)$ with the norm
$\|u\|_{1,p} = \|\nabla u\|_p$.

We will need the space

$$
W_0^{1,p(b(u))}(\Omega) = \{u \in W^{1,1}(\Omega)\} : \int_\Omega |\nabla u|^{p(b(u))}\, dx < \infty\}.
$$

Also, we introduce the functional space

$$X(Q)=\{u\in L^{\infty}(0,T;L^2(\Omega)):|\nabla u|\in L^{p(b(u))}(Q),u(.,t)\in V_t(\Omega)\quad \text{a.e. } t\in]0,T[\},$$

where

$$V_t(\Omega)=\{u\in L^2(\Omega)\}\cap W_0^{1,\alpha}(\Omega):|\nabla u|\in L^{p(b(u(.,t)))}(\Omega)\},$$

in which we will prove the existence of weak solutions for the nonlocalÂ´problem (1).

Theorem 2.1. *Provided that (2) and (3) hold together with $\alpha>2N/(N+2)$, $u_0\in L^2(\Omega)$,*
(A_0) $g:\mathbb{R}\to\mathbb{R}$ is a nondecreasing continuous function, surjective, with $g(0)=0$ and $|g(s)|\leq C|s|$ and $f\in L^{\alpha'}(Q)$, then (1) has a weak solution in $u\in X(Q)\cap C([0,T];L^2(\Omega))$.

Proof We will semi-discrete (1) in time t and solve the related nonlinear elliptic problem. Based on the semidiscrete problem, we construct the corresponding approximate solutions via a singular perturbation technique combined with the theory of Sobolev spaces with exponent variables and the Schauder fixed-point theorem. The key procedure is to establish necessary energy estimates for finding the limit of the approximate solutions via a compactness argument. ∎

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EXISTENCE AND ASYMPTOTIC PROPERTIES FOR A GENERALIZED LINEAR EVOLUTION EQUATION UNDER EFFECTS OF A LOGARITHMIC TYPE DISSIPATION

RUY C. CHARÃO & FÉLIX P. Q. GÓMEZ

[1]Federal University of Santa Catarina, UFSC, SC, Brazil, ruycharao@gmail.com
[2]Federal Technological University of Paraná, UTFPR, PR, Brazil, felix12gomez@gmail.com

Abstract

We consider a dissipation model of a logarithmic type to study a linear second order evolution equation. The associated Cauchy problem for this new model in $\mathbb{R}^n$ and study decay rates of solutions as $t \to \infty$ in L^2-sense. The operator L_θ considered in this paper was first introduced to dissipate the solutions of the wave equation in the paper studied by Charão-Ikehata [1]. We will discuss the asymptotic property of the solution as time goes to infinity to the linear Cauchy problem.

1 Introduction

We consider in this work a generalized type evolution equations under effects of a dissipative mechanism based on an operator L_θ, that combines the composition of logarithm function with the Laplace operator as follows,

$$\partial_t^2 u + (-\Delta)^\delta \partial_t^2 u + (-\Delta)^\alpha u + L_\theta \partial_t u = \beta(-\Delta)^\gamma (\partial_t u)^p, \quad (t,\ x) \in \,]0,\ \infty[\, \times \mathbb{R}^n, \tag{1}$$

$$u(0,\ x) = u_0(x), \quad \partial_t u(0,\ x) = u_1(x), \quad x \in \mathbb{R}^n, \tag{2}$$

where $u = u(t,\ x)$, $0 < \delta \leq \alpha$ with $\beta \neq 0$, $p > 1$ integer and the linear operator L_θ is as follows

$$L_\theta : D(L_\theta) \subsetneq L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \theta > 0,$$

with

$$D(L_\theta) := \left\{ f \in L^2(\mathbb{R}^n) : \int\limits_{\mathbb{R}^n} \log^2\left(1 + |\xi|^{2\theta}\right) \left|\widehat{f}(\xi)\right|^2 d\xi < +\infty \right\},$$

and for $f \in D(L_\theta)$,

$$L_\theta\, f(x) := \mathcal{F}^{-1}_{\xi \to x}\left[\log\left(1 + |\xi|^{2\theta}\right)\, \widehat{f}(\xi)\right] \qquad \text{that is,} \qquad \mathcal{F}[L_\theta f](\xi) = \log\left(1 + |\xi|^{2\theta}\right)\, \widehat{f}(\xi).$$

We note that the term $L_\theta \partial_t u$ in the equation (1) given by the logarithmic function is natural because logarithmic function appears in many natural phenomena.

Here, one has just denoted the Fourier transform $\mathcal{F}_{x\to\xi}[f](\xi)$ of $f(x)$ by

$$\mathcal{F}_{x\to\xi}[f](\xi) = \widehat{f}(\xi) := \frac{1}{(2\pi)^{1/n}} \int\limits_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

as usual with $i := \sqrt{-1}$ and $\mathcal{F}^{-1}_{\xi\to x}$ expresses its inverse Fourier transform. We also need the Sobolev Space,

$$\dot{H}^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{u}|^2\, d\xi < \infty \right\}.$$

Symbolically writing, one can see

$$L_\theta = \log\left(I + (-\Delta)^\theta\right),$$

where Δ is the usual Laplace operator defined on $H^2(\mathbb{R}^n)$.

2 Main Results

In order to introduce our main results we should define function spaces such that for $s \geq 0$

$$Y^s = \left\{ f \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n_\xi} \left(1 + |\xi|^2\right)^s |\widehat{f}(\xi)|^2 \, d\xi < \infty \right\}$$

with its natural norm

$$\|f\|_{Y^s} := \left(\int_{\mathbb{R}^n_\xi} \left(1 + |\xi|^2\right)^s \left|\widehat{f}(\xi)\right|^2 d\xi \right)^{1/2}, \quad f \in Y^s, \tag{3}$$

and its corresponding inner product. Then our results read sd follows.

Theorem 2.1 (Existence and Uniqueness). *Let $n \geq 1$, $0 \leq \theta \leq \delta \leq \alpha$ and $(u_0, \ u_1) \in Y^{2\alpha-\delta} \times Y^\alpha$. Then the problem (1)–(2) with $\beta = 0$ admits a unique solution in the class*

$$u \in C\left([0, \ \infty[\ ; \ Y^{2\alpha-\delta}\right) \cap C^1\left([0, \ \infty[\ ; \ Y^\alpha\right) \cap C^2\left([0, \ \infty[\ ; \ Y^\delta\right).$$

Moreover, for initial data $(u_0, \ u_1) \in X = Y^\alpha \times Y^\delta$ the problem (1)–(2) admits a unique weak solution in the class

$$u \in C\left([0, \ \infty[\ ; \ Y^\alpha\right) \cap C^1\left([0, \ \infty[\ ; \ Y^\delta\right) \cap C^2\left([0, \ \infty[\ ; \ L^2\right).$$

Proposition 2.1 (Asymptotic Behaviour). *Let $n \geq 3$, $\alpha > 2\theta$, $\delta > 0$ and let $u(t, \ \xi)$ be the solution to problem (1)–(2) with $\beta = 0$. Suppose that $u_0 \in L^1(\mathbb{R}^n) \cap \dot{Y}^{\delta\left(\frac{n-2\delta}{2\alpha-2\theta}\right)}$, $u_1 \in L^1(\mathbb{R}^n) \cap \dot{Y}^{\delta\left(\frac{n-2\delta}{2\alpha-2\theta}\right)-\alpha}$. Then, the following estimate holds,*

$$\int_{\mathbb{R}^n} |u(t, \ x)|^2 \, dx \leq C \, t^{-\frac{n-2\alpha}{2\alpha-2\theta}} \left[\|u_1\|^2_{L^1} + \|u_1\|^2_{\dot{Y}^{\delta\left(\frac{n-2\delta}{2\alpha-2\theta}\right)-\alpha}} + \|u_0\|^2_{\dot{Y}^{\delta\left(\frac{n-2\delta}{2\alpha-2\theta}\right)}} + \|u_0\|^2_{L^1} \right], \quad t > 0,$$

where C is positive constants depending only on n.

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REFINED DECAY RATES OF C_0-SEMIGROUPS ON BANACH SPACES

G. S. SANTANA & S. L. CARVALHO

[1]Universidade Estadual do Sudoeste da Bahia, DCET, BA, Brazil, genilson.santana@uesb.edu.br
[2]Universidade Federal de Minas Gerais, ICEX, MG, Brazil, silas2luiz@gmail.com

Abstract

We study rates of decay for C_0-semigroups on Banach spaces under the assumption that the norm of the resolvent of the semigroup generator grows with $|s|^{\beta}\log(|s|)^{b}$, $\beta, b \geq 0$, as $|s| \to \infty$, and with $|s|^{-\alpha}\log(1/|s|)^{a}$, $\alpha, a \geq 0$, as $|s| \to 0$. Our results do not suppose that the semigroup is bounded.

1 Introduction

An important question in the theory of differential equations refers to the asymptotic behavior (in time) of their solutions; more specifically, if they reach an equilibrium and, if so, with which speed. For those linear partial differential equations which can be conveniently analyzed by rewriting them as evolution equations, it is well known that the long-term behavior of the solutions of each one of these equations is related to some spectral properties (and behavior of the resolvent) of the generator of the associated semigroup.

The asymptotic theory of semigroups provides tools for investigating the convergence to zero of mild and classical solutions to the abstract Cauchy problem

$$\begin{cases} u'(t) + Au(t) = 0, & t \geq 0 \\ u(0) = x, \end{cases} \tag{1}$$

We know that (1) has a unique mild solution for every $x \in X$, and that the solution depends continuously on x if, and only if, $-A$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X.

The works of Lebeau [3] (and some others authors) raised the question of what is the relation between the growth rates for norm of the resolvent and the decay rates of the norm of semigroup orbits. More precisely, assuming a spectral condition under the generator, $\sigma(A) \subset \mathbb{C}_+$ in (1), and $\|R(is, A)\|_{\mathcal{L}(X)} \to \infty$ as $|s| \to \infty$, then $(T(t))_{t\geq 0}$ is not exponentially stable and one typically obtains other asymptotic behavior. Until 2010, much attention has been paid to polynomial decay rates of the norm of semigroup orbits. In the work of [2], BÃ¡tkai, Engel, Prüss and Schnaubelt proved that for uniformly bounded semigroups, a polynomial growth rate of the norm of the resolvent implies a specific polynomial decay rate for classical solutions to (1). more precisely, let $(T(t))_{t\geq 0}$ be a *bounded* semigroup on a Banach space X with infinitesimal generator $-A$ such that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $s \geq 0$ and set $M(s) := \sup_{|\xi|\leq s} \|(i\xi + A)^{-1}\|_{\mathcal{L}(X)}$. If there exist constants $C, \beta > 0$ such that $M(s) \leq C(1+s)^{\beta}$, then for each $\varepsilon > 0$, there exists a positive constant C_ε such that for each $t > 0$, $\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} \leq C_\varepsilon t^{-\frac{1}{\beta}+\varepsilon}$. In [1], Batty and Duyckaerts extended this correspondence to the case where the resolvent growth is arbitrary; they were also able to reduce the loss $\varepsilon > 0$ to a logarithmic scale: if $M : (0, \infty) \to (0, \infty)$ is a continuous non-decreasing function of positive increase such that $\|R(is, -A)\|_{\mathcal{L}(X)} \leq M(|s|)$; then, there exists a positive constant C such that $\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O\left(\frac{1}{M_{\log}^{-1}(Ct)}\right)$, $t \to \infty$, where $M_{\log}^{-1}$ is the right inverse of $M_{\log}(s) := M(s)(\log(1+M(s)) + \log(1+s))$. In particular, if $M(s) \leq C(1+s)^{\beta}$ for any $\beta > 0$ and $C > 0$, then $\|T(t)(1+A)^{-1}\|_{\mathcal{L}(X)} = O\left(\frac{\log(t)}{t}\right)^{1/\beta}$, $t \to \infty$. Until this point, we have presented some of the main results of the asymptotic theory of **bounded** C_0-semigroups. Nevertheless, there are many natural classes of examples where the norm of the resolvent of the generator grows with a power-law rate as $|s| \to \infty$, for example, but the semigroup is not uniformly bounded, or where it is unknown whether the semigroup is in fact bounded. To the best of our knowledge, the following result due to BÃ¡tkai, Engel, Prüss and Schnaubelt is the first in the literature that proves polynomial decay for not necessarily bounded semigroups. More precisely, let $(T(t))_{t\geq 0}$ be a semigroup defined in a Banach space X with generator $-A$ such that $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C_\beta(1+|\lambda|)^\beta$ with $\overline{\mathbb{C}_-} \subset \rho(A)$ and there exists $\beta > 0$ such that with $\lambda \in \overline{\mathbb{C}_-}$. Then, there exists a positive constant $C_{n,\delta}$ such that for each $n \in \mathbb{N}$, $\delta \in (0,1]$ and $t > 0$, $\|T(t)(1+A)^{-\beta(n+1)-1-\delta}\|_{\mathcal{L}(X)} \leq C_{n,\delta}t^{-n}$. Then, by using geometrical properties of the underlying Banach space (like its Fourier type), Rozendaal and Veraar have shown the following result (see Theorem 4.9 in [4]). Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator $-A$ defined in a Banach space X with Fourier type $p \in [1,2]$, and let $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Suppose that $\overline{\mathbb{C}_-} \subset \rho(A)$ and that there exist $\beta, C \geq 0$ such that $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C(1+|\lambda|)^\beta$ for each $\lambda \in \overline{\mathbb{C}_-}$. Let $\tau > \beta + 1$; then, for each $\rho \in \left[0, \frac{\tau - 1/r}{\beta} - 1\right)$, there exists $C_\rho \geq 0$ such that for each $t \geq 1$, $\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \leq C_\rho t^{-\rho}$.

2 Main Results

We have obtained decay rates for C_0-semigroups, by assuming that the norm of the resolvent of the generator behaves as a function of type $|s|^\beta \log(|s|)^b$ as $|s| \to \infty$ (a particular example of a *regularly varying* function). Under these assumptions on the resolvent and without the assumption of boundedness of the semigroup, to the best knowledge of the authors, these estimates are new. The proofs of the following Theorems can be found in our work [5].

Theorem 2.1. *Let $\beta > 0$, $b \geq 0$ and let $(T(t))_{t\geq 0}$ be a C_0-semigroup defined in the Banach space X with Fourier type $p \in [1,2]$, with $-A$ as its generator. Suppose that $\overline{\mathbb{C}_-} \subset \rho(A)$ and that for each $\lambda \in \mathbb{C}$ with $Re(\lambda) \leq 0$, $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \lesssim (1+|\lambda|)^\beta(\log(2+|\lambda|))^b$. Let $r \in [1,\infty]$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$, and let $\tau > 0$ be such that $\tau > \beta + \frac{1}{r}$. Then, for each $\delta > 0$, there exist constants $c_{\delta,\tau} \in [0,\infty)$ and $t_0 \geq 1$ such that for each $t \geq t_0$, $\|T(t)(1+A)^{-\tau}\|_{\mathcal{L}(X)} \leq c_{\delta,\tau} t^{1-\frac{\tau - r^{-1}}{\beta}} \log(1+t)^{\frac{b(\tau - r^{-1})}{\beta} + \frac{1+\delta}{r}}$.*

Theorem 2.2. *Let β, b, A, $(T(t))_{t\geq 0}$ and X be as in the statement of Theorem 2.1. Suppose A injective, $\overline{\mathbb{C}_-} \setminus \{0\} \subset \rho(A)$ and that there exist $\alpha \geq 1$, $\beta, a, b > 0$ and positive constants C_1 and C_2 such that $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C_1|\lambda|^{-\alpha}\log(1/|\lambda|)^a$, $|\lambda| \leq 1$, $\|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C_2|\lambda|^\beta \log(|\lambda|)^b$, $|\lambda| \geq 1$, with $\lambda \in \overline{\mathbb{C}_-} \setminus \{0\}$. Let σ, τ be such that $\sigma > \alpha - 1$ and $\tau > \beta + 1/r$. Then, for each $\rho \in \left[0, \min\left\{\frac{\sigma+1}{\alpha} - 1, \frac{\tau - r^{-1}}{\beta} - 1\right\}\right]$ and each $\delta > 1 - 1/r$, where $r \in [1,\infty]$ is such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{p'}$, there exist $C_{\delta,\rho} > 0$ and $t_0 \geq 1$ so that for each $t \geq 1$, $\|T(t)A^\sigma(1+A)^{-\sigma-\tau}\|_{\mathcal{L}(X)} \leq C_{\delta,\rho}t^{-\rho}\log(1+t)^{c(\lceil \rho \rceil + 1) + 1/r + \delta}$, with $c = \max\{a, b\}$.*

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ON A COUPLED SYSTEM OF THE NAVIER STOKES VOIGT TYPE

G. M. DE ARAÚJO

Universidade Federal do Pará/ICEN/FACMAT, UFPA, Pa, Brazil, gera@ufpa.br

Abstract

In this paper we investigate the problem for a model of the viscoelastic fluid system coupled. In our analysis we consider a fluid of the navier-stokes-voigt type. Thus, using the Faedo-Galerkin's approximations we establish our result on existence of weak solutions. Uniqueness of solutions is also analyzed.

1 Introduction

The equation below describe the motion of non-Newtonian fluid to which a small quantity of polymers is added:

$$\begin{aligned} u_t - \alpha\Delta u_t \quad &- \quad \nu\Delta u + (u\cdot\nabla)(u-\alpha\Delta u) + \nabla p = f \qquad \text{in } Q_T, \\ \nabla\cdot u \quad &= \quad 0 \text{ in } Q_T, u=0 \text{ on } \Sigma_T, u(0)=u_0 \text{ in } \Omega, \end{aligned} \tag{1}$$

where $u=(u_1,u_2,...,u_n)$ is the velocity, p represents the pressure, $f=(f_1,f_2,...,f_n)$ stands for the given external, $\nu>0$ is the constant kinematic viscosity parameter of the fluid and α is called the relaxation coefficient of fluid. The system (1) is known in the literature as Navier-Stokes-Voigt, as already been studied by A. P. Oskolkov [1, 2] and Amrouche [3], for example. We observe that if $\alpha=0$ in (1), we find the Navier-Stokes equations.
The equations that describes the motion of micropolar fluids are

$$\begin{gathered} u' - (\nu+\nu_r)\Delta u + (u\cdot\nabla)u + \nabla p = 2\nu_r \text{ rot } w + f \ \text{ in } Q_T, \\ w' - (c_a+c_d)\Delta w + (u\cdot\nabla)w - (c_o+c_d-c_a)\nabla(\nabla\cdot w) + 4\nu_r w, = 2\nu_r \text{ rot } u + g, \text{ in } Q_T, \\ \nabla\cdot u = 0 \quad \text{in} \quad Q_T, u=0 \quad \text{on} \quad \Sigma_T, w=0 \quad \text{on} \quad \Sigma_T, \end{gathered} \tag{2}$$

where $u(x,t)$, $w(x,t)$ and $p(x,t)$, denotes, respectively, the unknown velocity, the micro-rotational velocity and the hydrostatic pressure of the fluid. The constants ν and ν_r are, respectively, the *Newtonian* and *micro-rotational viscosity*; the positive constants c_0, c_a and c_d are called *coefficients of angular viscosities* and satisfies $c_0+c_d>c_a$. These systems have been mainly analyzed in the book of *G. Lukaszewicz* [4].
In the present work, we propose a similar problem to that of G. Lukaszewicz (2), for the fluid Naver-Stokes-Voigt type, where we establish existence ($n=3$) and uniqueness ($n=2$) theorems:

$$\begin{gathered} u_t - \alpha\Delta u_t - (\nu+\nu_r)\Delta u + (u\cdot\nabla)\,u + \nabla p = 2\nu_r \text{rot } w + f \text{ in } Q_T, \\ w_t - \Delta w_t - \nu_1\Delta w + (u\cdot\nabla)\,w - \nu_2\nabla(\nabla\cdot w) + 4\nu_r w = 2\nu_r\text{rot } u + g \text{ in } Q_T, \\ \nabla\cdot u = 0 \ \text{ in } \ Q_T, u=0 \ \text{ on } \ \Sigma_T, w=0 \ \text{ on } \ \Sigma_T, u(x,0)=u_0(x) \ \text{ in } \ \Omega, w(x,0)=w_0(x) \ \text{ in } \ \Omega, \end{gathered} \tag{3}$$

2 Main Results

Definition 2.1. *We suppose $n=3, u_0\in H$, $w_0\in L^2(\Omega)^3$, $f, f_t\in L^2(0,T;H)$ and $g, g_t\in L^2(0,T;L^2(\Omega)^3$. A weak solution to the boundary value problem (3) is a pair of functions $\{u,w\}$, such that $u\in L^2(0,T;V)\cap L^\infty(0,T;H)$, $w\in L^2(0,T;H_0^1(\Omega)^3)\cap L^\infty(0,T;L^2(\Omega)^3)$, for $T>0$, satisfying the identity*

$$\begin{gathered} (u_t,\varphi) + \alpha a(u_t,\varphi) + (\nu+\nu_r)a(u,\varphi) + b(u,u,\varphi) = 2\nu_r(rot\ w,\varphi) + (f,\varphi) \\ (w_t,\phi) + a(w_t,\varphi) + \nu_1 a(w,\phi) + b(u,w,\phi) + \nu_2(div\ w, div\ \phi) + 4\nu_r(w,\varphi) = (rot\ u,\phi) + (g,\phi)\ \forall\,\varphi\in V,\ \phi\in H_0^1(\Omega)^3 \\ \nabla\cdot u = 0,\ \ u(0)=u_0,\ \ w(0)=w_0. \end{gathered} \tag{4}$$

Theorem 2.1. *If $f, f_t \in L^2(0,T;H)$, $g, g_t \in L^2(0,T : L^2(\Omega)^3$, $u_0 \in V$ and $w_0 \in H_0^1(\Omega)^3$, then there exists a pair of functions $\{u, w\}$ defined for $(x,t) \in Q_T$, solution to the boundary value problem (3) in the sense of Definition 2.1.*

Idea of the proof In the first estimate we multiply the approximate equation associated with $(4)_1$ and equation $(4)_2$ by u^m and w^m respectively. We obtain the following estimates

$$u^m \text{ is bounded in } L^\infty(0,T;V) \text{ and } w^m \text{ is bounded in } L^\infty(0,T;H_0^1(\Omega)^3) \tag{5}$$

In the second estimate we take the derivative with respect t only in the approximate equation associated with $(4)_1$ and multiply by u_t^m and the approximate equation associated with $(4)_2$ we multiply by w_t^m. We obtain the following estimates

$$(w_t^m) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)^3) \text{ and } (u_t^m) \text{ is bounded in } L^\infty(0,T;V). \tag{6}$$

It follows from the above estimates that

$$u_i^m \to u_i \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ and a.e. in } Q \text{ and } w_i^m \to w_i \text{ strongly in } L^2(0,T;L^2(\Omega)) \text{ and a.e. in } Q,$$

$$\text{therefore, } u_i^m u_j^m \to u_i u_j \text{ a.e. in } Q \text{ and } w_i^m w_j^m \to w_i w_j \text{ a.e. in } Q. \tag{7}$$

$$\text{If } n = 3, \text{ we have } u_i^m \in L^4(0,T;H_0^1(\Omega)) \hookrightarrow L^4(0,T;L^4(\Omega)), \text{ then}$$

$$\|u_{m_i} u_{m_j}\|^2_{L^2(0,T;L^2(\Omega))} \leq C, \text{ therefore}$$

$$u_i^m u_j^m \to u_i u_j \quad \text{weak in } \; L^2(0,T;L^2(\Omega)).$$

Follows from the previous convergences and the Lebesgue dominated convergence theorem that

$$(u_j^m \varphi_i)^2 \to (u_j \varphi_i)^2 \text{ strong in } L^1(Q), \text{ therefore}$$

$$(u_j^m \varphi_i) \to (u_j \varphi_i) \text{ strong in } L^2(Q).$$

Using the convergences above, we conclude that

$$b(u^m, u^m, \phi) = -b(u_m, \phi, u^m) = -\sum_{i,j=1}^n \int_\Omega u_i^m \frac{\partial \phi_j}{\partial x_i} u_j^m dx \to -\sum_{i,j=1}^n \int_\Omega u_i \frac{\partial \phi_j}{\partial x_i} u_j dx = b(u,u,\phi).$$

$$b(u^m, w^m, \phi) = \sum_{i,j=1}^n \int_\Omega u_i^m \frac{\partial w_j^m}{\partial x_i} \phi_j dx \to \sum_{i,j=1}^n \int_\Omega u_i \frac{\partial w_j}{\partial x_i} \phi_j dx = b(u,w,\phi).$$

Uniqueness follows from the method of energy.

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UNIQUENESS OF ENTROPY SOLUTION FOR DOUBLE NONLINEAR ISOTROPIC DEGENERATE FRACTIONAL PARABOLIC PROBLEM

G. HUAROTO

Universidade Federal de Alagos,IM, AL, Brazil, gerardo.cardendas@im.ufal.br

Abstract

In this paper, we investigate the uniqueness of solutions for the double nonlinear isotropic degenerate fractional parabolic problem within bounded domains, subject to homogeneous. To this end, we double variables as Kružkov [1] to obtain this contraction. Since this technique is by now well understood for scalar conservation laws in bounded domains,

1 Introduction

We study in this paper the existence of solutions for the following initial-boundary value problem

$$\begin{cases} \partial_t u + \mathtt{div}\,\mathbf{f}(u) + (-\Delta)^s_{\Omega,p} A(u) = 0 & \text{in } \Omega_T, \\ u|_{t=0} = u_0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 & \text{on } \Gamma_T, \end{cases} \tag{1}$$

where

$$(-\Delta)_p u = -\mathtt{div}(|\nabla u|^{p-2}\nabla u),$$

and $u : \Omega_T \to \mathbb{R}$ is the unknown function that is sought, $\Omega_T := (0,T)\times\Omega$, $\Gamma_T = (0,T)\times\Gamma$ for any real number $T > 0$ and $\Omega \subset \mathbb{R}^d$ is a bounded open set having smooth (C^2) boundary Γ. Moreover, the flux function $\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$ is in $C^1(\mathbb{R})$ and $\mathbf{f}'$ is locally Lipschitz and $A \in C^1(\mathbb{R})$ is a nondecreasing function and A' is locally Lipschitz (without loss of generality $A(0) = 0$). The initial data $u_0 \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ and $(-\Delta)^s_{\Omega,p}$ is the Regional Fractional p-Laplacian, that is to say

$$(-\Delta)^s_{\Omega,p} A(u) = \lim_{\epsilon\to 0} C_{d,s,p} \int_{\Omega\setminus B_\epsilon(x)} |A(u)(x) - A(u)(y)|^{p-2} \frac{A(u)(x) - A(u)(y)}{|x-y|^{d+ps}} dy$$

where $C_{d,s,p}$ is the normalized constant (see [2]).

Different particular cases of this problem have been studied by several researchers. The author, Andreianov and Neves [6] investigated the existence of solution for $p = 2$, while Cifani and Jakobsen [9] studied the problem over the entire space. Other important work includes that of Wei, Duan, and LV [3], where they study the existence of kinetic solution. Furthermore, the extremal cases of $s = 1$ and $p = 2$ have been thoroughly researched, with important work. Mascia, Porretta, and Terracina [5], Michel and Vovelle [4]. Other case, in which $p > 1$, has received considerable attention from researchers. Bendahmane and Karlsen's work on establishing the uniqueness of entropy solutions is an important contribution to this field [8]. Additionally, Igbida and Urbano's research in the context of multiple dimensions is noteworthy [7]. Another notable situation arises when $A' = 0$. In this specific case, the problem (1) takes on the characteristics of a well-known nonlinear hyperbolic problem, which has been extensively investigated by Kružkov [1].

2 Main Results

The aim of this section is to establish the uniqueness of weak entropy solution for the isotropic degenerate equations

$$\partial_t u + \mathtt{div}\,\mathbf{f}(u) + (-\Delta)^s_{\Omega,p} A(u) = 0,$$

posed in $Q_T = (0,T)\times\Omega$, and for any $p \in (1,\infty)$ and $\max\{\frac{1}{p^\star}, \frac{1}{p}\} < s < 1$. Our main result is the following theorem

Theorem 2.1. *Let u and v be two entropy solutions of (1), with initial data u_0, $v_0 \in L^\infty(\Omega)$ respectively. Then for a.e. $t \in (0,T)$*

$$\int_\Omega |u(t,x) - v(t,x)|^+ dx \leq \int_\Omega |u_0(x) - v_0(x)|^+ dx.$$

Consequently $\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\Omega)} \leq \|u_0 - v_0\|_{L^1(\Omega)}$. If $u_0 \leq v_0$ a.e. in Ω, then $u \leq v$ a.e. in Q_T. Finally, if $u_0 = v_0$ a.e. in Ω, then $u = v$ a.e in Q_T.

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PERIODIC SOLUTIONS OF THE SECOND ORDER DYNAMIC EQUATIONS ON TIME SCALES AND APPLICATIONS

M. CHOQUEHUANCA[1], T. CUCHTA[2], & J. G. MESQUITA[3]

[1]University of La Frontera, Temuco, Chile, mario.choquehuanca@ufrontera.cl
[2]Marshall University, Huntington, WV 25755, USA, cuchta@marshall.edu
[3]Universidade de Brasília, Brasília-DF, Brazil, jgmesquita@unb.br

Abstract

We investigate the existence of solutions of linear and semilinear second-order equations involving time scales. To obtain such results, we make use of the equivalence between the second-order equations with its self-adjoint equation and fixed point results. Also, we present some examples and applications to illustrate our main results.

1 Introduction

It is known in the real world that continuously varying processes can be modeled by differential equations, and that processes that vary discretely can be modeled by difference equations. But there are other processes that vary both continuously and discreetly, which was a challenge for mathematicians, but in 1988 Stefan Hilger in his Ph. D. dissertation [1], achieves start the continuous and discrete analysis, giving in this way, the beginning of the theory of calculus in time scales and dynamic equations on time scales (see [2, 3] and references cited therein), its main features being those of unification and extension, providing us with a powerful tool to deal with such mixed processes.

The theory of periodic functions has achieved a considerable development so far [4, 5, 6, 7], but the investigations related to the existence of periodic solutions in dynamic second-order equations on time scales are still scarce. Motivated by these facts, we focus our attention to investigate in this paper the semilinear and linear second order dynamic equations on time scales, respectively, given by:

$$x^{\Delta\Delta}(t) = A(t)x^{\Delta}(t) + B(t)x(t) + g(t), \quad t \in \mathbb{T}, \tag{1}$$

and

$$x^{\Delta\Delta}(t) = A(t)x^{\Delta}(t) + B(t)x(t) + f(t, x(t)), \quad t \in \mathbb{T}, \tag{2}$$

where $A, B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, $g \in \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}^n)$ and $f \in \mathcal{C}_{rd}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n)$. Also, we assume that A, B, g and f are ω-periodic functions on $\mathbb{T}$ and $C(t) = A^T(t) - \mu(t)B^T(t)$ such that it is satisfied

$$e_C(t, s) \text{ and } (\ominus C)(t) \text{ commute.} \tag{3}$$

$$\Phi(t) = [\ominus (C)] (t) \tag{4}$$

2 Main Results

The following three results are related to the existence of $\omega-$periodic solutions of equation (1).
Let $P_\omega = \{\varphi \in C_{rd}(\mathbb{T}, \mathbb{R}^n) : \varphi(t + \omega) = \varphi(t)\}$ be and $\|\varphi\| = \sup_{t\in\mathbb{T}} |\varphi(t)|$ for $\varphi \in P_\omega$.

Lemma 2.1. *Let $\mathbb{T}$ be a $\omega-$periodic time scale and assume that A, B and g are $\omega-$periodic on $\mathbb{T}$. Then*

$$e_\Phi(t,s)\Phi(t) = \Phi(t)e_\Phi(t,s), \ \textit{for } t,s \in \mathbb{T}, \tag{1}$$

$$K_\Phi := (e_\Phi(t+\omega,t)^T - I)^{-1}, \tag{2}$$

is independent of $t \in \mathbb{T}$ whenever $\Phi \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$. Furthermore, if $x \in P_\omega$, we have $x^\Delta \in P_\omega$.

Lemma 2.2. *Let $\mathbb{T}$ be a $\omega-$periodic time scale and assume that A, B and g are $\omega-$periodic on $\mathbb{T}$. If $x \in P_\omega$, then x is a solution of following equation*

$$\left(e_\Phi(t,t_0)^T x^\Delta(t)\right)^\Delta - e^\sigma_\Phi(t,t_0)^T B(t)x^\sigma(t) = e^\sigma_\Phi(t,t_0)^T g(t), \tag{3}$$

if, and only if

$$x(t) = K_\Phi \int_t^{t+\omega} \left\{ e_\Phi(\tau,t)^T \Phi^T(\tau)x^\sigma(\tau) + \int_{-\infty}^{\tau} e_\Phi(\sigma(s),t)^T G(s,x^\sigma(s))\,\Delta s \right\} \Delta\tau, \tag{4}$$

where Φ, K_Φ are given by (4) and (2) respectively and $G(t,y) := B(t)y + g(t)$ for $y \in \mathbb{R}^n$.

Theorem 2.1. *Let $\mathbb{T}$ be a $\omega-$periodic time scale and assume that A, B and g are $\omega-$periodic on $\mathbb{T}$. Assume further that $G(t,x) := B(t)x+g(t)$ is bounded with respect to both variables on $\mathbb{T}\times\mathbb{R}^n$, that is there exists a positive constant M such that*

$$\|G(t,x)\| \le M, \quad (t,x) \in \mathbb{T}\times\mathbb{R}^n.$$

Then equation (3) has a $\omega-$peridic solution.

The following theorem is related with the existence of $\omega-$periodic solutions of equation (2)

Theorem 2.2. *Let $\mathbb{T}$ be a $\omega-$periodic time scale and assume that A, B are $\omega-$periodic on $\mathbb{T}$. Assume further that f is ω-periodic with respect to the first variable and that there exists constants, $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le L\|x-y\|, \quad x,y \in \mathbb{R}^n, \ t \in \mathbb{T}, \tag{5}$$

Then equation

$$\left(e_\Phi(t,t_0)^T x^\Delta(t)\right)^\Delta - e^\sigma_\Phi(t,t_0)^T B(t)x^\sigma(t) = e^\sigma_\Phi(t,t_0)^T f(t,x(t)), \tag{6}$$

has a $\omega-$periodic solution.

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HENSTOCK KURZWEIL INTEGRAL AND APPLICATIONS

A. K. A. SILVA

Universidade de Brasília, UnB, DF, Brazil, aryel.kath@gmail.com

Abstract

A function $f \to \mathbb{R}^n$ is said to be *Henstock-Kurzweil Integrable* on I if there exists a vector $B \in \mathbb{R}^n$ such that for every $\epsilon > 0$, there exists a gauge γ_ϵ on I such that if $\dot{P} := (I_i, t_i)_{i=1}^n$ is any tagged partition of I with $l(I_i) < \gamma_\epsilon(t_i)$ for $i = 1, ..., n$, then

$$||S(f; \dot{P}) - B|| \leq \epsilon$$

The model proposed by Henstock began with the investigation of an integration process aimed at reconstructing a derived function and is responsible for encompassing a broader class of functions than those in the Riemann and Lebesgue models, without the need to work with measure theory as in Lebesgue integrable functions. At the same time, but completely independently, Jaroslav Kurzweil introduced an equivalent integration concept in 1957 to investigate results of continuous dependence.

This type of integration naturally pays more attention to taggings than the more traditional integration concepts. Therefore, the definition is constructed by allowing the $\gamma_\epsilon > 0$ used in the Riemann integral definition to be any positive function, which allows a broader class of functions to be integrable.

Using this type of integral, it is possible to study many important problems in physics with highly oscillatory behavior, such as the Kapitza pendulum.

1 Introduction

The concept of the integral arises from the attempt to calculate areas and volumes of figures, and one of the techniques employed is precisely approximation by known figures. Over time, it becomes evident that the process of integration also has a strong connection with differentiation.

Its development goes through Riemann in the 1850s, who separates these concepts again using limits and summations, and is equivalent to the concept presented by Darboux when working with bounded functions, which uses the concept of upper and lower integrals of a bounded function over an interval. Thus, when considering all functions over an interval where the integration process could be defined, we have:

A function $f \to \mathbb{R}$ is said to be *Riemann-integrable* on I if there exists a number $A \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists a $\gamma_\epsilon > 0$, such that if $\dot{P} := (I_i, t_i)_i^n = 1$ is any tagged partition of I, where $l(I_i) < \gamma_\epsilon(t_i)$, with $i = 1, ..., n$, then we have

$$|S(f; \dot{P}) - A| \leq \epsilon$$

By the early 20th century, Lebesgue proposed a new concept of the integral, more general and capable of integrating a larger number of functions, which solves several problems related to integrals, such as the validity of the Fundamental Theorem of Calculus. According to Lebesgue, for the Fundamental Theorem to be valid, it is necessary for the function to have a bounded derivative. From this, it was natural to question a new concept of the integral, where if f is integrable and differentiable according to this concept, then its derivative f' is also integrable, and the Fundamental Theorem holds. The development of this problem gives rise to the Generalized Riemann Integral, or Henstock-Kurzweil Integral, a subject of study in this work, which presents a simpler and more general formulation than the Lebesgue Integral.

2 Main Results

The Henstock-Kurzweil integral naturally focuses more on taggings than traditional integration models. Thus, this concept is built by allowing the $\gamma_\epsilon > 0$ used in the Riemann definition to be any positive function. This permits a broader class of functions to be integrable. Such $\gamma_\epsilon > 0$ is called a *gauge*, and we have the following definitions found in [1]:

Definition 1

If $I : [a, b] \subset \mathbb{R}$, a function $\delta \to \mathbb{R}$ is a gauge on I_s and $\delta(t) > 0$ for all $t \in I$. The interval around $t \in I$ is controlled by the gauge δ in the interval:

$$B[t, \delta(t)] := [t - \delta(t), t + \delta(t)]$$

Definition 2

If $I \subset [a, b]$ is an interval and $\dot{P} = (I_i, t_i)_i^n = 1$ is a tagged subpartition. If δ is a gauge on I, $\dot{P}$ is δ-fine; then for all $i = 1, ..., n$:

$$I_i \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$$

From there, considering the definitions of gauge functions, we have that the Henstock-Kurzweil Integral is given by the following definition:

Definition 3

A function $f \to \mathbb{R}$ is said to be Henstock-Kurzweil-integrable on I if there exists a number $B \in \mathbb{R}$ such that for every $\epsilon > 0$, there exists a gauge function γ_ϵ on I such that if $\dot{P} := (I_i, t_i)_i^n = 1$ is any tagged partition of I, where $l(I_i) < \gamma_\epsilon(t_i)$, with $i = 1, ..., n$, then we have

$$|S(f; \dot{P}) - B| \leq \epsilon$$

The existence of the gauge function in the definition of the Henstock-Kurzweil integral motivates its generality and is the main difference compared to the Riemann integral.

3 Conclusion

Advances in integration theory were driven by attempts to generalize the integral concept addressed by Riemann and Lebesgue. While some methods used the Lebesgue integral as a particular case, Henstock started by investigating an integration process aimed at reconstructing the function using the derivative, employing the concepts of Riemann and Darboux. This process, which we refer to as the Henstock-Kurzweil Integral or Generalized Riemann Integral, encompasses a broader class of functions than those covered by Riemann and Lebesgue, without the need to use measure theory as required for Lebesgue integrable functions.

In this project, the focus was on the study of the Henstock-Kurzweil Integral, also known as the new integration theory, as well as its properties and specifications for the Fundamental Theorem of Calculus, starting from differentiated integration models such as the Riemann Integral.

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DECREASING AND EXPONENTIAL STABILITY FOR GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

E. TOON & F. ANDRADE DA SILVA

[1]Universidade Federal de Juiz de Fora, DM-ICE, MG, Brasil, eduard.toon@ufjf.br
[2]Universidade de São Paulo, ICMC, SP, Brasil, ffeandrade@gmail.com

Abstract

The theory of generalized ordinary differential equations (generalized ODEs, for short) is a very powerful theory once several types of equations can be regarded as them. In this lecture, we aim present a new concept of stability, which we call decreasing stability, and deal with some Lyapunov techniques on decreasing and exponential stability.

1 Introduction

A. M. Lyapunov, in 1892, developed two methods for analyzing the stability of differential equations. One can consider direct and converse Lyapunov Theorems. A direct Lyapunov Theorem states that if a Lyapunov functional exists then an equilibrium point is stable. On the other hand, a converse Lyapunov Theorem claims that if an equilibrium point is stable then a Lyapunov functional exists.

In what concerns generalized ordinary differential equations (we write generalized ODEs for short), the study of stability has recently increased. This is due the fact that generalized ODEs encompass several types of equations, as for instance, functional differential equations of neural type, measure functional differential equations, dynamic equations on time scales, integral equations and a class of partial differential equations. See [2].

Motivated by these facts, we are interested in presenting stability criteria for generalized ODEs. At first, we introduce two new concepts of stability, called decreasing stability and asymptotic stability, which generalize exponential stability (we will show this during the lecture). This lecture is based on the submitted paper [1].

Definition 1.1. *Let $s_0 \geq t_0 \geq 0$, $x_0 \in X$ and $x : [s_0, +\infty) \to X$ be the global forward solution of the generalized ODE*

$$\frac{dx}{d\tau} = DF(x,t), \tag{1}$$

with initial condition $x(s_0) = x_0$. The trivial solution of the generalized ODE (1) is called

1. *exponentially stable, if there exist positive constants ρ, α, β such that*

$$\|x(t)\| = \|x(t, s_0, x_0)\| < \alpha e^{-\beta(t-s_0)}, \quad \text{for all } t \in [s_0, +\infty),$$

whenever $\|x_0\| < \rho$;

2. *decreasingly stable, if there exist $\delta > 0$ and a decreasing function $\sigma : [0, +\infty) \to \mathbb{R}^+$ such that $\sigma(0) < \infty$ and, if $\|x_0\| < \delta$, then $\|x(t)\| = \|x(t, s_0, x_0)\| < \sigma(t - s_0)$ for all $t \in [s_0, +\infty)$;*

3. *decreasingly asymptotically stable, if it is decreasingly stable if $\sigma(t) \to 0$ as $t \to \infty$.*

Furthermore, we will present a Lyapunov-type Theorem in the context of decreasing stabilities and we obtain converse Lyapunov Theorems for these stabilities and for exponential stability.

2 Main Results

Our main results are

Theorem 2.1. *If there exists a functional* $V : [t_0, +\infty) \times X \to \mathbb{R}$ *such that*

1. $V(t, z) \geq 0$ *for every* $(t, z) \in [t_0, +\infty) \times X$*;*
2. *the mapping* $[s_0, +\infty) \ni t \mapsto V(t, x(t))$ *is nonincreasing along every solution* $x : [s_0, +\infty) \to X$ *of the generalized ODE (1);*
3. *there exists a positive constant* γ *such that* $\gamma\|z\| \leq V(t, z)$ *for all* $(t, z) \in [t_0, +\infty) \times X$*;*

then the trivial solution of the generalized ODE (1) is decreasingly stable.

Corollary 2.1. *If there exists a functional* $V : [t_0, +\infty) \times X \to \mathbb{R}$ *satisfying conditions 1, 2 and 3 from Theorem 2.1 such that the mean of* V *is zero along every maximal solution, that is,*

$$\lim_{t\to\infty} \frac{1}{t - s_0} \int_{s_0}^{t} V(\tau, x(\tau))d\tau = 0,$$

where $x : [s_0, +\infty) \to X$ *is a solution of the generalized ODE (1). Then, the trivial solution of the generalized ODE (1) is decreasingly asymptotically stable.*

Theorem 2.2. *If the trivial solution of the generalized ODE (1) is decreasingly stable, then there exist* $\delta > 0$ *and a functional* $V : [t_0, +\infty) \times B_\delta \to \mathbb{R}^+$*,* $B_\delta = \{x \in X; \|x\| < \delta\}$*, such that*

1. $V(t, y) \geq 0$ *for all* $(t, y) \in [t_0, +\infty) \times B_\delta$*;*
2. $V(\cdot, y) : [t_0, +\infty) \to \mathbb{R}^+$ *is left-continuous on* $(t_0, +\infty)$ *for all* $y \in B_\delta$*;*
3. *there exists a monotonically increasing and continuous function* $a : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\|y\| \leq V(t, y) \leq a(\|y\|)$ *for all* $(t, y) \in [t_0, +\infty) \times B_\delta$*;*
4. *the mapping* $[s_0, +\infty) \ni t \mapsto V(t, x(t))$ *is nonincreasing along every solution* $x : [s_0, +\infty) \to B_\delta$ *of the generalized ODE (1).*

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ON STOCHASTIC STABILIZATION VIA CONTROL LYAPUNOV FUNCTIONALS

F. ANDRADE DA SILVA

Universidade de São Paulo, São Carlos, Brazil, ffeandrade@gmail.com

Abstract

Generalized stochastic equations (GSEs) are described by the Kurzweil-belated integral which extends the Itô-Henstock integral to include a larger class of equations. This enables one to deal with highly oscillatory (operator valued) functions of unbounded variation. The main goal of this work is concerned with stabilization of nonlinear dynamical systems by means of Lyapunov functionals.

1 Introduction

As the workhorse of modern analysis, integrals are, without question, among the most familiar calculus tools. The first integral that comes to our mind when we think about calculus is the Riemann integral. However, this integral has several limitations and the class of Riemann integrable functions is quite limited. To overcome these limitations, Henri Lebesgue used a considerably amount of measure theory to create a new notion of integral. Decades later, independently, Ralph Henstock (1955) and Jaroslav Kurzweil (1957) came up with a new and much simpler formulation of integral which encompasses the Riemann, Newton and Lebesgue integrals. In their definition, the intuitive approach of the Riemann integral is preserved, but unlike the latter, Henstock and Kurzweil considered a strictly positive function δ (called gauge) to calibrate the length of each subintervals of the domain. By making this "small adjustment", it turned out that their integral, known as the Henstock-Kurzweil integral, recovers all primitives as integrals. Moreover, this integral allows us to deal with highly oscillating integrands.

Usually, in stochastic calculus, the integrators and the integrands are highly oscillatory. Therefore, it is impossible to define stochastic integrals using the Riemann or Lebesgue absolute integration theories. One well-known stochastic integral, namely the Itô integral, reminds us the Lebesgue approach of integrable functions through elementary ones. The concept of Itô-Henstock integral, on the other hand, has been studied since 1969 and reduces technicalities in the classical way of defining stochastic integrals.

Generalized ordinary differential equations are described by the Henstock-Kurzweil integral and they have been shown to act as a unifying theory for many equations. In order to create a similar environment to the non-deterministic case, the authors in [4] defined a new class of equations, called generalized stochastic equations (we write GSEs for short), whose solutions are described by the Kurzweil-belated integral. The idea of this integral is to use belated partial divisions and adapt the classic Kurzweil integral so that it not only it contains the Itô-Henstock integral but it also provides a general setting for many stochastic equations.

It is well-known that stability conditions for solutions of differential equations can be obtained using an appropriate Lyapunov functional. Moreover, the construction of different Lyapunov functionals allows obtaining different stability conditions and the reciprocal is true. Stability theory for GSEs in the framework of Lyapunov functionals can be found in [1, 2]. Lyapunov theorems have a long history and have been frequently used in nonlinear control problems to establish robustness of asymptotic stability. Indeed, control Lyapunov function is a central tool in stabilization and generalizes an abstract energy function-a Lyapunov functional-to the case of controlled systems.

This paper aims to provide a general framework for stabilization by means of nonsmooth control Lyapunov functionals using sampled controls. Stabilization here was meant as practical stabilization, i.e., the system state could be stabilized into any desirable vicinity of the equilibrium, provided that the sampling time be sufficiently small. Our main result generalizes the related results to the stochastic case [3].

2 Generalized Stochastic Equations

In this section, we provide the basic background on GSEs introduced in [4].

Definition 2.1. *Let $\delta\colon [a,b] \to [0,+\infty)$ be a non-negative function (called gauge on $[a,b]$). A δ-belated partial division of $[a,b]$ is any finite collection of point-interval pairs, $D = \{(x_{i-1},(x_{i-1},x_i]) : i = 1,2,\ldots,|D|\}$, such that $(x_{i-1},x_i]$, $i = 1,2\ldots,|D|$, are disjoint left-open subintervals of $[a,b]$ and $(x_{i-1},x_i] \subset (x_{i-1},x_{i-1}+\delta(x_{i-1}))$, for all $i = 1,2\ldots,|D|$. If, in addition, for a given $\eta > 0$, $\left|b-a-\sum_{i-1}^{|D|}(x_i - x_{i-1})\right| \leq \eta$, then $D = \{(x_{i-1},(x_{i-1},x_i]) : i = 1,2,\ldots,|D|\}$, is called a (δ,η)-belated partial division of $[a,b]$, that is, D fails to cover $[a,b]$ by at most a set o Lebesgue measure η.*

Definition 2.2. *Let $\mathfrak{F}(\Omega,V)$ be the space of all operators from Ω to a Hilbert space V. Assume that $G\colon [a,b]\times[a,b] \to \mathfrak{F}(\Omega,V)$ is a $\{\mathcal{F}_t\}$-adapted process on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in I},\mathbb{P})$. We say that G is Kurzweil-belated integrable over $[a,b]$, if for every $\epsilon > 0$, there exist an element $K \in L^p(\Omega,V)$, a gauge δ on $[a,b]$ and $\eta > 0$ such that $\int_\Omega \left\|\sum_{i=1}^{|D|}[G(s_{i-1},s_i) - G(s_{i-1},s_{i-1})](\omega) - K(\omega)\right\|_V^p d\mathbb{P} < \epsilon$, for every (δ,η)-fine belated partial division $D = \{(s_{i-1},(s_{i-1},s_i]) : i = 1,2\ldots,|D|\}$ of $[a,b]$. In this case, we write $K = \int_a^b G(\tau,s)$.*

Definition 2.3. *Let $F\colon L^p(\Omega,V)\times J \to \mathfrak{F}(\Omega,V)$. A $\{\mathcal{F}_t\}$-adapted process $X = \{X_t : t \in J\}$ on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\in I},\mathbb{P})$, with $X_t \in L^p(\Omega,V)$, for all $t \in J$, is a solution of the GSE*

$$X_t = X_s + \int_s^t F(X_r,\tau), \quad t,s \in J, \tag{1}$$

on J, whenever $X_t(\omega) \in V$ for every $t \in J$ and $\mathbb{P}$-almost every $\omega \in \Omega$ and the integral equation (1) holds, where the integral is in the sense of the Kurzweil-belated integral with $G(r,\tau) = F(X_r,\tau)$.

Definition 2.4. *A functional $V\colon [t_0,+\infty)\times L^p(\Omega,V) \to \mathbb{R}^+$ is said to be positive definite (in the sense of Lyapunov) if $V(t,0) \equiv 0$ and, for some $\mu \in \mathcal{K}$, $V(t,Z) \geq \mu(\|Z\|_{L^p})$, for all $(t,Z) \in [t_0,+\infty)\times L^p(\Omega,V)$. On the other hand, V is said to be negative definite, if $-V$ is positive definite.*

3 Main Results

Theorem 3.1. *Consider the GSE (1) and suppose there exists a control Lyapunov functional $V\colon [t_0,+\infty)\times L^p(\Omega,V) \to \mathbb{R}^+$ such that $c_1\mathbb{E}[\|X_t\|_V^p] \leq V(t,X_t) \leq c_2\mathbb{E}[\|X_t\|_V^p]$ for some positive constants c_1,c_2 and all $t \geq s_0$. Then, there is a unique solution of (1) for each $\widetilde{X} \in L^p(\Omega,V)$ and the trivial solution is globally stable in probability.*

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OXYGEN DIFFUSION MODEL IN A CELL DESCRIBED USING A FRACTIONAL OPERATOR

S. R. MAZORCHE & T. F. C. SANTOS

[1]Universidade Federal de Juiz de Fora, ICE-DM, MG, Brazil,
sandro.mazorche@ufjf.br, tiago.santos@estudante.ufjf.br

Abstract

This paper presents a fractional model of oxygen diffusion in cellular tissue. We will start with a technique that proposes an approximate problem, as seen in [2]. Through the ideas described in the papers [5, 6], we will introduce a fractional operator in time using the Riemann-Liouville fractional derivative. A numerical study will be presented with these approximate solutions, focusing on formulating this fractional model as a nonlinear complementarity problem. We will use the FDA-NCP algorithm to obtain the numerical solution of the model and the curve representing the moving boundary.

1 Introduction

The oxygen diffusion model is a classic problem and has been studied for over 50 years. We will briefly describe the physical part of the model and present the equations that describe it in dimensionless form. More details can be found in [1, 2]. The problem consists of determining the concentration $c(x,t)$ at each instant and monitoring the movement of the boundary that delimits the presence of oxygen in the tissue, known as the moving boundary, $S_0(t)$.

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} - 1\,,\ 0 \le x \le S_0(t)\,,\ t \ge 0. \tag{1}$$

$$c = \frac{\partial c}{\partial x} = 0\,,\ x = S_0(t)\,,\ t \ge 0\,; \tag{2}$$

$$\frac{\partial c}{\partial x} = 0\,,\ x = 0\,,\ t \ge 0\,; \tag{3}$$

$$c = \frac{1}{2}(1-x)^2\,,\ 0 \le x \le 1\,,\ t = 0. \tag{4}$$

Our proposal for constructing a fractional model is based on the approach presented in [5, 6]. Thus, we will use the kernel $K(t) = e^{-r_1 t} E_\alpha(-r_2^\alpha t^\alpha)$, with $r_1 + r_2 = 1$ where the function $E_\alpha(z)$ is the one-parameter Mittag-Leffler function. After calculations similar to [5] we define a fractional model by replacing equation (1.1) with

$$\frac{\partial c}{\partial t} = {}^*D^\alpha \left[\frac{\partial^2 c}{\partial x^2}\right] - 1\ ,\ \text{and keeping equations (1.2)-(1.4).} \tag{5}$$

Where ${}^*D^\alpha[y(t)] = r_1 y(t) + r_2^\alpha e^{-r_1 t}\ {}^{RL}D_t^{1-\alpha}\left[e^{r_1 t} y(t)\right]$ and ${}^{RL}D_t^{1-\alpha}$ is the Riemann-Liouville derivative

$${}^{RL}D_t^{1-\alpha}[y(t)] = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \frac{y(\tau)}{(t-\tau)^{1-\alpha}}d\tau\,,\ 0 < \alpha \le 1. \tag{6}$$

As done in [2], the solution is given by

$$c(x,t) = a_0(t) + \sum_{k=1}^{\infty} a_k(t)\cos(k\pi(1-x)). \tag{7}$$

2 Main Results

Replacing (7) in the fractional model (1.5) we obtain:

$$c(x,t) = \frac{1}{6} - t + \frac{2}{\pi^2}\sum_{k=1}^{\infty}(-1)^k e^{-r_{1k}t} E_\alpha(-r_{2k}^\alpha t^\alpha)\frac{\cos(k\pi(1-x))}{k^2} \quad \text{, where } r_{1k} + r_{2k} = k^2\pi^2. \tag{8}$$

The complementarity problem version is presented below, as discussed in [4]. The FDA-NCP algorithm [3], was used to determine the numerical solution($C \equiv c(x,t)$).

$$\frac{\partial C}{\partial t} - {}^*D^\alpha\left[\frac{\partial^2 C}{\partial x^2}\right] + 1 \geq 0 \quad , \quad C \geq 0 \quad \text{and} \quad \left(\frac{\partial C}{\partial t} - {}^*D^\alpha\left[\frac{\partial^2 C}{\partial x^2}\right] + 1\right) C = 0. \tag{9}$$

We use the values $\alpha = 0.8$, $r_{1k} = 0.4(k\pi)^2$ and $r_{2k} = 0.6(k\pi)^2$ to generate Figures $1(a)-(b)$. In Figure $1(a)$, we see the surface generated by the solution given by equation (8) where we can see the intersection of the surface with the XoY plane that defines the free boundary curve. In Figure $1(b)$, we see the solution $c(x,t)$ found by the FDA-NCP algorithm, of the complementarity problem (9) for the following time values: $t = 0$ (black, initial condition), $t = 0.03$ (red), $t = 0.06$ (blue), $t = 0.09$ (green), $t = 0.12$ (cyan), $t = 0.15$ (yellow).

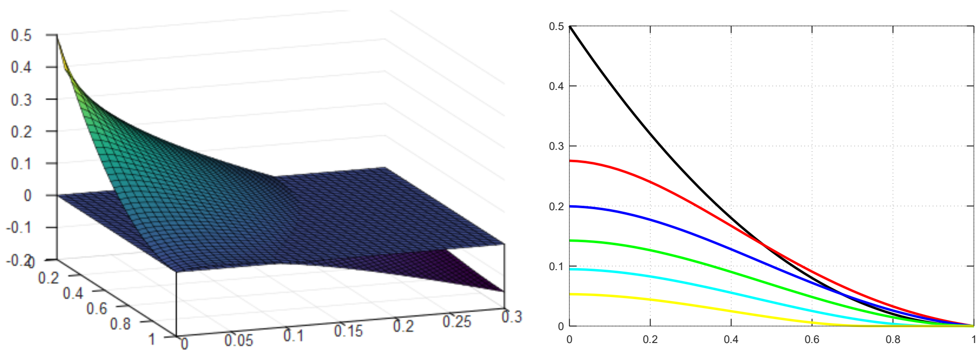

Figure 1: (a)-(b)

The results obtained are promising and studies on the effects of this fractional operator in the model are already advanced. We hope that this work contributes to the study of fractional models.

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A NONLOCAL EQUATION OF $P(U)$-LAPLACIAN TYPE

GABRIEL RODRíGUEZ V.[1], EUGENIO CABANILLAS L.[1], WILLY BARAHONA M.[1], LUIS MACHA C.[1] & ROCíO DE LA CRUZ M.[1]

[1]Universidad Nacional Mayor de San Marcos, GI-EDOACBI. FCM, Lima - Perú,
grodriguezv@unmsm.edu.pe, cleugenio@yahoo.com, grodriguezv@unmsm.edu.pe,
lmachac@unmsm.edu.pe, rdelacruzm@unmsm.edu.pe

Abstract

The object of this work is to study the existence of solutions for a nonlocal $p(u)$-Laplacian Dirichlet problem with a nonlocal nonlinearity. Firstly, we establish our results applying the degree theory for (S_+) type mappings together with the technique of Zhikov for passing to the limit in a sequence of $p(u_n)$-Laplacian problems, then we conclude our result by using a fixed point theorem.

1 Introduction

This research is devoted to the study of the following nonlocal $p(u)$-Laplacian problem

$$\begin{aligned} -\mathcal{A}(u)\text{div}(|\nabla u|^{M\left(\int_\Omega |\nabla u|^2\,dx\right)-2}\nabla u) &= \mathcal{B}(u)f(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\mathcal{A}, \mathcal{B}$ are two functionals defined on a Sobolev sapce, M and f are functions that satisfy conditions which will be stated later. In [1, 5, 3] the authors consider the problem (1), with $\mathcal{A}(u) = 1 = \mathcal{B}(u)$, $M\left(\int_\Omega |\nabla u|^2\,dx\right) = p(b(u))$ with p, b continuous functions and $f(x,u) = f(x)$. They showed existence of solutions via the theory of monotone operators and the Brouwer fixed-point theorem. Motivated by the above references, we deal with the existence of solutions for nonlocal $p(u)$-Laplacian problem(1), but we are facing serious difficulties when we want to apply the theory of monotone operators in the Banach space $W^{1,p(b(\cdot))}(\Omega)$ to prove the existence of weak solutions. Therefore, we use the degree theory for (S_+) type mappings to obtain some useful convergence results and the approximation procedure employed by Zhikov [4]. Next, the mentioned fixed point theorem allows us to conclude our result.

2 Notations and Main Results

We note that $M\left(\int_\Omega |\nabla u|^2\,dx\right)$ is here a real number and not a function, then the Sobolev spaces involved in this work are the classical ones. We will consider the well-known Sobolev space $W_0^{1,p}(\Omega)$ with the norm
$\|u\|_{1,p} = \|\nabla u\|_p$. Assume that the following assumptions hold:

(M_0) $M : [0,+\infty[\to]m_0, m_1[$ is a continuous function with $m_0 > 0$.

(A_0) $\mathcal{A} : X \to [0,+\infty[$, $\mathcal{B} : X \to \mathbb{R}$ are continuous and bounded on any bounded subset of $X = W_0^{1,M\left(\int_\Omega |\nabla u|^2\,dx\right)}(\Omega)$, with $\mathcal{A}(0) > 0$ for all $u \in X \setminus \{0\}$ and for any bounded sequence $\{u_\nu\} \subset X$ for which $\mathcal{A}(u_\nu) \to 0$ we have $u_\nu \to 0$ in X.

(F_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory and there exists η with $1 < \eta < m_1 - 1$ such that

$$|f(x,u)| \le c_1 + c_2|u|^\eta \quad \text{a.e. } x \in \Omega,\ \text{all } u \in \mathbb{R}, c_1, c_2 > 0$$

Theorem 2.1. *Let* (M_0), (A_0) , (F_1) *and the following conditions hold:*

(A_1) *There are constants* $\alpha \in \mathbb{R}$, $M > 0$ *and* $c_3 > 0$ *such that* $\mathcal{A}(u) \geq c_3 \|u\|^{\alpha}$ *all* $u \in X$ *with* $\|u\| \geq M$.

(B_1) *There are constants* $\beta \in \mathbb{R}$, $M > 0$ *and* $c_4 > 0$ *such that* $|\mathcal{B}(u)| \leq c_4 \|u\|^{\beta}$ *all* $u \in X$ *with* $\|u\| \geq M$.

(H_1) $\alpha + m_0 > \beta + \eta$ *and* $\alpha + m_0 > 0$

Then (1) *has a weak solution in* X.

Proof We employ the degree theory for (S_+) type mappings combined with the technique of Zhikov, and the Brouwer fixed-point theorem. ■

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ASYMPTOTICS FOR SOBOLEV EXTREMALS: THE HYPERDIFFUSIVE CASE

GREY ERCOLE

Universidade Federal de Minas Gerais, ICEx, MG, Brazil, grey.ercole@gmail.com

Abstract

Let Ω be a bounded, smooth domain of $\mathbb{R}^N$, $N \geq 2$. For $p > N$ and $1 \leq q(p) < \infty$ set

$$\lambda_{q(p)} := \inf \left\{ \int_\Omega |\nabla u|^p \, dx : u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega |u|^{q(p)} \, dx = 1 \right\}$$

and let $u_{q(p)}$ denote a corresponding positive extremal function. We prove that if $\lim_{p\to\infty} \frac{q(p)}{p} = \infty$, then each sequence $u_{q(p_n)}$, with $p_n \to \infty$, admits a subsequence converging uniformly in $\overline{\Omega}$ to a viscosity solution to the problem

$$\begin{cases} -\Delta_\infty u = 0 & \text{in} \quad \Omega \setminus M \\ u = 0 & \text{on} \quad \partial\Omega \\ u = 1 & \text{in} \quad M, \end{cases}$$

where M is a closed subset of the set of all maximum points of the distance function to the boundary of Ω.

1 Introduction

We study the asymptotic behavior, as $p \to \infty$, of the pair $\left(\lambda_{q(p)}, u_{q(p)}\right)$ where

$$\lambda_{q(p)} := \inf \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega) \text{ and } \|u\|_{q(p)} = 1 \right\}$$

and $u_{q(p)}$ is a positive minimizer, which is also a weak solution to the Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda_{q(p)} |u|^{q(p)-2} u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

(Ω is a smooth, bounded domain of $\mathbb{R}^N$, $N \geq 2$).

The introductory case $q(p) = p$ was studied by Juutinen, Lindqvist and Manfredi [3]. They first showed that

$$\lim_{p\to\infty} \lambda_p^{1/p} = \Lambda_\infty := \|d_\Omega\|_\infty^{-1}$$

where d_Ω denotes the distance function to the boundary of Ω. Then, they proved that any sequence u_{p_n}, with $p_n \to \infty$, admits a subsequence that converges uniformly in $\overline{\Omega}$ to a viscosity solution $u_\infty \in C_0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ to the Dirichlet problem

$$\begin{cases} \min\left\{|\nabla u| - \Lambda_\infty u, -\Delta_\infty u\right\} = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Such a function is known as a first eigenfunction of the ∞-Laplacian.

Charro and Peral in [2] and Charro and Parini in [1], studied the asymptotic behavior, as $p \to \infty$, of the positive weak solutions u_p to the problem

$$\begin{cases} -\Delta_p u = \mu_p |u|^{q(p)-2} u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

under the assumption that $\mu_p > 0$ is such that

$$\Lambda := \lim_{p\to\infty} \mu_p^{1/p} \in (0,\infty).$$

Let us set

$$Q := \lim_{p\to\infty} \frac{q(p)}{p}.$$

In [2] it is considered the *subdiffusive* case: $Q \in (0,1)$ whereas in [1] it is considered the *superdiffusive* case: $Q \in (1,\infty)$. In both works it is proved that any sequence u_{p_n}, with $p_n \to \infty$, admits a subsequence converging uniformly to a viscosity solution to the problem

$$\begin{cases} \min\left\{|\nabla u| - \Lambda u^Q, -\Delta_\infty u\right\} = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

2 Main Results

Theorem 2.1. *If $\lim_{p\to\infty} q(p) = \infty$, then*

$$\lim_{p\to\infty} \lambda_{q(p)}^{1/p} = \Lambda_\infty \quad \textit{and} \quad \lim_{p\to\infty} \left\|u_{q(p)}\right\|_\infty = 1.$$

Moreover, each sequence $u_{q(p_n)}$, with $p_n \to \infty$, admits a subsequence that converges uniformly to a function $u_\infty \in C_0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ which enjoys the following properties:

1. $0 \le u_\infty(x) \le \Lambda_\infty d_\Omega(x)$ *for all* $x \in \overline{\Omega}$.
2. $\|u_\infty\|_\infty = 1$ *and* $\Lambda_\infty = \|\nabla u_\infty\|_\infty$.
3. $M := \{x \in \Omega : u_\infty(x) = 1\} \subseteq M_\Omega := \{x \in \Omega : d_\Omega(x) = \|d_\Omega\|_\infty\}$.
4. u_∞ *is infinity superharmonic in* Ω *and (consequently) positive in* Ω.

Corollary 2.1. *If $Q = 1$, then the function $u_\infty \in C_0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ obtained in Theorem 1 is a first eigenfunction to the ∞-Laplacian.*

Our main result, stated in the sequence, focuses on the limit problem satisfied by the limit function u_∞ in the case not yet treated in the literature, which we call *hyperdiffusive* case: $Q = \infty$.

Theorem 2.2. *If $Q = \infty$, then the function $u_\infty \in C_0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ obtained in Theorem 1 is a viscosity solution to the problem*

$$\begin{cases} -\Delta_\infty u = 0 & \textit{in} \quad \Omega \setminus M \\ u = 0 & \textit{on} \quad \partial\Omega \\ u = 1 & \textit{in} \quad M. \end{cases}$$

Moreover, $u_\infty = \frac{d_\Omega}{\|d_\Omega\|_\infty}$ if and only if $M = M_\Omega = \Sigma_\Omega := \{x \in \Omega : d_\Omega \text{ is not differentiable at } x\}$.

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VARIABLE SUPERCRITICAL SCHRÖDINGER-POISSON SYSTEM WITH SINGULAR TERM

ANDERSON L. A. DE ARAUJO[1], LUIZ F. O. FARIA[2], & JEFERSON C. SILVA[2]

[1] Dep. Mathematics, Federal University of Viçosa, Viçosa - MG, Brazil, anderson.araujo@ufv.br

[2]Dep. Mathematics, Federal University of Juiz de Fora, Campus Universitário, Juiz de Fora - MG, Brazil, luiz.faria@ufjf.br, jefersonbs2009@gmail.com

Abstract

In this paper, we consider a class of Schrödinger-Poisson systems on bounded domains that depend on a parameter η. These systems have variable supercritical exponents and a singular term as part of their nonlinearity. For $\eta = 1$, we proved the existence and uniqueness of a solution using variational methods. In the case $\eta = -1$, the structure of the problem changes significantly, and we proved the existence of a solution using non-variational methods based on an approximating scheme. In both cases, we faced difficulties handling the loss of compactness because the variable exponents involve supercritical growth. The supercritical variable growth not only causes the system to lose its homogeneity but also its compactness properties.

1 Introduction

Consider the following system

$$\begin{cases} -\Delta u + u + q\phi f(u) = g(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 2F(u), & \text{in } \mathbb{R}^3. \end{cases} \tag{1}$$

This type of system has been the subject of intensive research because of its strong application relevance. From a physical point of view, it describes systems of identically charged particles interacting with each other in cases where magnetic effects can be neglected. The nonlinear term $g(x,u)$ models the interaction between the particles, while the coupled term $\phi f(u)$ concerns the interaction with the electric field (see [1, 2] for details). System (1), also known as the nonlinear Schrödinger-Maxwell problem, was proposed in [3] as a model describing solitary waves for the nonlinear stationary Schrödinger equations interacting with the electrostatic field.

Recent research has focused on the existence, nonexistence, multiplicity results, and ground state or sign-changing solutions of system (1) under various assumptions on the nonlocal term f and the nonlinear term g.

The Schrödinger-Poisson system in a bounded domain has also been extensively studied in the literature. In this article, we study a class of Schrödinger-Poisson systems in a bounded domain with variable supercritical exponents and a singular term. More precisely, we consider the following problem:

$$\begin{cases} -\Delta u + 2(2^* + r^\alpha - 1)\,\eta\,\phi u^{2^*+r^\alpha-2} & = \frac{\lambda}{u^\gamma} \quad \text{in } \Omega, \\ -\Delta\phi & = u^{2^*+r^\alpha-1} \quad \text{in } \Omega, \\ u > 0, \quad \phi > 0 & \quad\quad \text{in } \Omega, \\ u & = \phi = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is the ball in $\mathbb{R}^N$ centered at the origin and of radius one, $N \geq 3$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $r = |x|$, $\eta = \pm 1$, $\alpha, \lambda > 0$ are real parameters, and $\gamma \in (0,1)$.

Schrödinger-Poisson systems with a singular term have garnered significant attention recently due to the added complexities introduced by this term, such as challenges in analyzing the behavior of solutions near singularities and the non-smoothness of associated energy functionals. For example, the energy functional associated with the system is not of class C^1, prompting researchers to seek new techniques for obtaining solutions via critical point theory. In [4], the authors considered the following Schrödinger-Poisson system with singularity and critical exponent:

$$\begin{cases} -\Delta u + \eta\,\phi u^{2^*-2} & = \frac{\lambda}{u^\gamma} \quad \text{in } \Omega, \\ -\Delta\phi & = u^{2^*-1} \quad \text{in } \Omega, \\ u > 0, & \phi > 0 \quad \text{in } \Omega, \\ u & = \phi = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $\eta = \pm 1$, $\lambda > 0$ is a real parameter, and $\gamma \in (0,1)$. For $\eta = 1$, the authors demonstrated that system (3) has a unique solution, and for $\eta = -1$, they demonstrated that system (3) has at least two different solutions. There are fewer articles on the matter when considering the supercritical exponent in Sobolev's sense. Nonetheless, in a recent paper [5], do Ó *et al.* studied a second-order elliptic equation involving variable supercritical exponents in the unit ball, achieving an unexpected result of existence.

As far as we know, quasilinear Schrödinger-Poisson systems with singular terms and variable supercritical exponents have not been studied. Motivated by the references mentioned above, specifically [4] and [5], we propose in this article to study the problem of obtaining a solution for system (1), which has a singular term and variable supercritical exponent. Besides the fact that we cannot directly apply critical points theory to study the energy functional associated with system (1), the second equation of the system has a term with variable supercritical growth. This term presents a significant difficulty in control, as it not only lacks compactness due to its growth but also loses any type of homogeneity in this equation.

2 Main Results

The main results of this article are given by the following theorems whose proof can be found in [6].

Theorem 2.1. *If $\eta = 1$, then for each $\lambda > 0$, system (1) has a unique radial solution $(u,\phi) \in H_0^1(\Omega) \times W_0^{2,\frac{2^*+1}{2^*}}(\Omega)$.*

Theorem 2.2. *If $\eta = -1$, then there exists $\Lambda > 0$ such that for all $\lambda \in (0,\Lambda)$, system (1) has at least one radial solution $(u,\phi) \in H_0^1(\Omega) \times W_0^{2,\frac{2^*+1}{2^*}}(\Omega)$.*

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ON A PLANAR EQUATION INVOLVING $(2, Q)$-LAPLACIAN WITH ZERO MASS AND TRUDINGER-MOSER NONLINEARITY

J. A. CARDOSO[1], J. C. DE ALBUQUERQUE[2], J. CARVALHO[1] & G. M. FIGUEIREDO[3]

[1]Universidade Federal de Sergipe, DMA, SE, Brazil, anderson@mat.ufs.br, jonison@mat.ufs.br
[2]Universidade Federal de Pernambuco, DMAT, PE, Brazil, josecarlos.melojunior@ufpe.br
[3]Universidade de Brasília, MAT, DF, Brazil, giovany@unb.br

Abstract

In this work, we study existence of positive solutions to a class of $(2, q)$-equations in the zero mass case in $\mathbb{R}^2$. We establish weighted Sobolev embedding and we introduce a new Trudinger-Moser type inequality. Moreover, since we work on a suitable radial Sobolev space, we prove a version of the Symmetric Criticality Principle. Finally, we study regularity of solutions applying Moser iteration scheme.

1 Introduction

We consider a class of $(2, q)$-equations in the zero mass case, namely

$$-\Delta u - \Delta_q u = Q(|x|)f(u), \quad \text{in } \mathbb{R}^2, \tag{$\mathcal{P}$}$$

where $1 < q < 2$, $Q : (0, \infty) \to \mathbb{R}$ is a radially symmetric weight function and

(Q) is continuous, $Q > 0$, and there exist $b_0, b \in \mathbb{R}$ such that

$$\limsup_{r\to 0^+} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r\to\infty} \frac{Q(r)}{r^{b}} < \infty.$$

Inspired by [1], we consider the space E^q, defined as the completion of $C_0^\infty(\mathbb{R}^2)$ with respect to the norm

$$\|u\|_{E^q} := \left(\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u\|^2_{L^q(\mathbb{R}^2)}\right)^{1/2}.$$

We also consider the space $E^q_{\text{rad}} := \{u \in E^q : u \text{ is radial}\}$ with the norm induced by E^q. Moreover, for $1 \le p < \infty$, we define the weighted Lebesgue space $L^p_Q(\mathbb{R}^2) := \left\{u : \mathbb{R}^2 \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^2} Q(|x|)|u|^p dx < \infty\right\}$, with norm

$$\|u\|_{L^p_Q(\mathbb{R}^2)} := \left(\int_{\mathbb{R}^2} Q(|x|)|u|^p \,\mathrm{d}x\right)^{1/p}.$$

Regarding the nonlinearity f, we consider the following assumptions:

(f_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exists $\alpha_0 > 0$ such that $\lim_{|s|\to\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \text{if } \alpha > \alpha_0, \\ \infty, & \text{if } \alpha < \alpha_0; \end{cases}$

(f_2) $f(s) = o(|s|^{\widetilde{q}-1})$, as $s \to 0$, where $\widetilde{q} := \max\left\{q^*, q^*\left(\frac{b}{2} + 1\right)\right\}$, with $q^* := 2q/(2 - q)$;

(f_3) there exists $\mu > \widetilde{q}$ such that, for any $s \neq 0$, we have $0 < \mu F(s) := \mu \int_0^s f(t)\,\mathrm{d}t \le f(s)s$;

(f_4) there exist $\xi > 0$ and $\nu > \widetilde{q}$ such that, for any $s \in (0, 1]$, one has $F(s) \ge \xi s^\nu$.

Remark 1.1. *We suppose that $f(s) = 0$, for $s \le 0$. An example satisfying our assumptions is $F(s) = |s|^\nu e^{\alpha_0 s^2}$ for $s > 0$ and $F(s) = 0$ for $s < 0$.*

2 Main Results

Theorem 2.1 (**Weighted Sobolev embedding**). *Assume that* (Q) *holds with* $b_0, b > -2$. *Then, the embedding* $E^q_{\mathrm{rad}} \hookrightarrow L^p_Q(\mathbb{R}^2)$ *is continuous for* $\widetilde{q} \le p < \infty$. *Furthermore, the embedding* $E^q_{\mathrm{rad}} \hookrightarrow L^p_Q(\mathbb{R}^2)$ *is compact, for* $\widetilde{q} < p < \infty$.

In the next proposition, we clarify more precisely where the embedding holds or not.

Proposition 2.1. *Suppose that* (Q) *holds.*

(i) If $b < -2$, *then* E^q_{rad} *is not continuously embedded into* $L^p_Q(\mathbb{R}^2)$ *for* $0 < p < \infty$;

(ii) If $b > -2$, *then* E^q_{rad} *is not continuously embedded into* $L^p_Q(\mathbb{R}^2)$ *for* $0 < p < q^*(b/2+1)$;

(iii) If $b > -2$, *then the embedding* $E^q_{\mathrm{rad}} \hookrightarrow L^p_Q(\mathbb{R}^2)$ *is continuous for* $q^*(b/2+1) < p < \infty$.

Remark 2.1. *In view of Theorem 2.1 and Proposition 2.1 (iii), if* $b \ge 0$, *then* $q^*(b/2+1)$ *is a sharp exponent to the Sobolev embedding.*

Consider the function $\Phi_\alpha(s) := e^{\alpha s^2} - \sum_{j=0}^{j_0-1} \frac{\alpha^j}{j!} s^{2j}$, where $s \in \mathbb{R}$, $\alpha > 0$, and $j_0 := \inf\{j \in \mathbb{N} : j \ge \widetilde{q}/2\}$.

Theorem 2.2 (**Sharp Trudinger-Moser type inequality**). *Suppose that* (Q) *holds with* $b_0, b > -2$. *For each* $\alpha > 0$ *and* $u \in E^q_{\mathrm{rad}}$, *we have that* $Q(|\cdot|)\Phi_\alpha(u) \in L^1(\mathbb{R}^2)$. *Furthermore,*

$$L(\alpha, Q) := \sup_{\{u \in E_{\mathrm{rad}} : \|u\|_{E^q} \le 1\}} \int_{\mathbb{R}^2} Q(|x|)\Phi_\alpha(u)\,\mathrm{d}x < \infty,$$

whenever $0 < \alpha \le 4\pi(b_0/2+1)$. *In addition, if*

$$\liminf_{r \to 0^+} \frac{Q(r)}{r^{b_0}} > 0, \tag{1}$$

then $L(\alpha, Q) = \infty$, *when* $\alpha > 4\pi(b_0/2+1)$.

Theorem 2.3 (**Existence of weak solutions**). *Assume that* (Q) *holds with* $b_0, b > -2$ *and* $(f_1)-(f_4)$ *hold. There exists* $\xi_0 > 0$ *such that if* (f_4) *holds for* $\xi \ge \xi_0$, *then Problem (1) admits a non-negative weak solution* $u \in E^q_{\mathrm{rad}} \setminus \{0\}$.

Remark 2.2. *(**Non-existence result**) The condition* $b > -2$ *is strongly necessary to obtain weak solutions for (1).*

We are also interested in studying regularity and positivity of weak solutions.

Theorem 2.4. *Assume the conditions of Theorem 2.3 and let* u *be a weak solution of (1). Then:*

(i) If (1) holds, then $u \in L^\infty(\mathbb{R}^2)$;

(ii) If (1) holds and $Q(|\cdot|) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$, *then* u *is positive and belongs to* $C^{1,\sigma}_{\mathrm{loc}}(\mathbb{R}^2)$, *for some* $\sigma \in (0,1)$;

(iii) $\left(\nabla u + |\nabla u|^{q-2}\nabla u\right) \in C^1(\mathbb{R}^2 \setminus \{0\})$ *and* u *solves (1) pointwise in* $\mathbb{R}^2 \setminus \{0\}$.

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EXISTENCE OF POSITIVE AND NONNEGATIVE EIGENFUNCTIONS FOR A FOURTH ORDER OPERATOR WITH DEFINITE AND INDEFINITE WEIGHTS

J. P. PINHEIRO DA SILVA

Universidade Federal do Pará - UFPA, ICEN, PA, Brazil, jpabloufpa@gmail.com

Abstract

In this work, we study the existence of solutions for the following eigenvalue problem:

$$(\mathcal{LP}) \quad \begin{cases} (-\Delta + d_1)(-\Delta + d_2)u + m(x)u = \lambda a(x)u & \text{in } \Omega \\ u \not\equiv 0, \ u \geq 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $d_1, d_2 \in \mathbb{R}$, and $a(\cdot), m(\cdot) \in L^\infty(\Omega)$ may have indefinite sign.

1 Introduction

Let $a(\cdot)$, $m(\cdot) \in L^\infty(\Omega)$ and $\mathcal{L}u := (-\Delta + d_1)(-\Delta + d_2)u + m(x)u$. In this paper, we shall be concerned with the existence of solution for the following linear eigenvalue problem:

$$(\mathcal{LP}) \quad \begin{cases} \mathcal{L}u = \lambda a(x)u & \text{in } \Omega \\ u \not\equiv 0, \ u \geq 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $N \geq 1$ and $d_1, d_2 \in \mathbb{R}$. Here and after $\lambda_1 := \lambda_1(\Omega)$ represents the first eigenvalue for $(-\Delta, H_0^1(\Omega))$, and the functions $m(\cdot)$ and $a(\cdot)$ may have an indefinite sign. By solution, we mean a function $u \in W^{4,2}(\Omega) \cap C^3(\overline{\Omega})$ that satisfies $(\mathcal{LP})$.

Weighted eigenvalue problems have extensive applications in various fields, including engineering, physics, and applied mathematics. Notably, they find relevance in the study of transport theory, reaction-diffusion equations, fluid dynamics, and selection-migration models in population dynamics, among others. Over the past four decades, there has been a growing interest in solving eigenvalue problems of the following form:

$$\begin{cases} Lu := -\displaystyle\sum_{j,k=1}^{N} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_j(x)\frac{\partial u}{\partial x_j} + a_0(x)u = \lambda a(x)u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this equation, $\Omega \subset \mathbb{R}^N$ represents a smooth domain, and L is a strongly uniformly elliptic differential operator of second order. The coefficient functions $a_{jk}(\cdot) = a_{kj}(\cdot)$, $a_j(\cdot)$, and $a_0(\cdot)$ are real-valued and non-negative, while $a(\cdot)$ is a given function that may change sign. The parameter $\lambda \in \mathbb{R}$ is an eigenvalue.

Despite the substantial literature on second-order operators, few works have addressed weighted eigenvalue problems involving fourth-order operators or higher-order operators under Navier boundary condition ($\Delta u = u = 0$ on $\partial\Omega$ for fourth-order operators) or even other boundary conditions.

As far as we know, the work [2, Proposition 2.3] is the only one in the literature that establishes the existence of a positive eigenfunction for the problem $(\mathcal{LP})$ with $m(\cdot) \equiv 0$ and $d_1, d_2 > \lambda_1(\Omega)$. However, the condition imposed on $a(\cdot)$ is extremely restrictive.

In this work, we present a result on the existence of a nonnegative eigenfunction and also results on the existence of a positive solution for a fourth-order operator with Navier boundary conditions and a weight function that can have an indefinite sign. It is worth noting that we demonstrate that the Krein-Rutman theory is not sufficient to solve this type of problem.

2 Main Results

Theorem 2.1. *Let* $\mu^* = \min_{n\in\mathbb{N}}(\lambda_n + d_1)(\lambda_n + d_2)$, $a(\cdot) \in L^\infty(\Omega) \setminus \{0\}$ *satisfying* $a(x) \geq 0$ *in* Ω *and* $m(\cdot) \in L^\infty(\Omega)$ *satisfying* $m(x) \leq 0$ *in* Ω. *If the following hypotheses are valid:*

(i) $d_1 + d_2 > -2\lambda_1$;

(ii) there exists $c_1 > 0$ *such that* $c_1 a(x) + m(x) + \mu^* \geq 0$ *in* Ω.

Then the linear problem below admits solution in $W_0^{1,2}(\Omega) \cap W^{4,2}(\Omega) \cap C^3(\overline{\Omega})$

$$(\mathcal{LP}_{12}) \qquad \begin{cases} (-\Delta + d_1)(-\Delta + d_2)u + m(x)u = \lambda a(x)u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda = \lambda_{12}(a)$ *is given by* $\lambda_{12}(a) := \inf\left\{ \dfrac{I_m(\varphi)}{J_a(\varphi)} \;\middle|\; \varphi \in H \text{ and } J_a(\varphi) \neq 0 \right\} > -\infty$, *with*

$$I_m(\varphi) := \|\varphi\|^2 + (d_1 + d_2)|\nabla\varphi|_2^2 + d_1 d_2 |\varphi|_2^2 + \int_\Omega m(x)\varphi^2 \quad \text{and} \quad J_a(\varphi) := \int_\Omega a(x)\varphi^2 dx.$$

Theorem 2.2. *Suppose* $N > 4$, $a(\cdot), m(\cdot) \in L^\infty(\Omega)$, $a^+ \not\equiv 0$, $d_1, d_2 > -\lambda_1$, *and* $\|m^-\|_\infty < (\lambda_1 + d_1)(\lambda_1 + d_2)$. *Under these conditions, the problem below admits solution in* $W_0^{1,2}(\Omega) \cap W^{4,2}(\Omega) \cap C^3(\overline{\Omega})$

$$(\mathcal{LP}_4) \qquad \begin{cases} (-\Delta + d_1)(-\Delta + d_2)u + m(x)u = \lambda_{ma1} a(x)u & \text{in } \Omega \\ u \not\equiv 0,\ u \geq 0 & \text{in } \Omega \\ \Delta u = u = 0 & \text{on } \partial\Omega \end{cases}$$

where λ_{ma1} *is given by* $0 < \dfrac{1}{\lambda_{ma1}} = \sup\left\{ \dfrac{\int_\Omega a(x)\varphi^2 dx}{\|\varphi\|_*^2 + \int_\Omega m(x)\varphi^2 dx} \;\middle|\; \varphi \in H \setminus \{0\} \right\}$.

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STEIN-WEISS PROBLEMS VIA NONLINEAR RAYLEIGH QUOTIENT FOR CONCAVE-CONVEX NONLINEARITIES

EDCARLOS D. SILVA[1], M. L. M. CARVALHO[1] & MÁRCIA S. B. A. CARDOSO[1]

[1]Universidade Federal de Goiás,GO,Brazil
edcarlos@ufg.br, marcos-leandro-carvalho@ufg.br, marcia.cardoso@ifg.edu.br

Abstract

In the present work, we consider existence and multiplicity of solutions for nonlocal elliptic problems driven by the Stein-Weiss problem with concave-convex nonlinearities defined in the whole space $\mathbb{R}^N$. More precisely, we consider the following nonlocal elliptic problem:

$$-\Delta u + V(x)u = \lambda a(x)|u|^{q-2}u + \int_{\mathbb{R}^N} \frac{b(y)|u(y)|^p dy}{|x|^\alpha |x-y|^\mu |y|^\alpha} b(x)|u|^{p-2}u, \text{ in } \mathbb{R}^N, \ u \in H^1(\mathbb{R}^N),$$

where $\lambda > 0, \alpha \in (0, N), N \geq 3, 0 < \mu < N, 0 < \mu + 2\alpha < N$. Furthermore, we assume also that $V : \mathbb{R}^N \to \mathbb{R}$ is a bounded potential, $a \in L^r(\mathbb{R}^N), a > 0$ in $\mathbb{R}^N$ and $b \in L^t(\mathbb{R}^N), b > 0$ in $\mathbb{R}^N$ for some specific $r, t > 1$. We assume also that $1 \leq q < 2$ and $2_{\alpha,\mu} < p < 2^*_{\alpha,\mu}$ where $2_{\alpha,\mu} = (2N - 2\alpha - \mu)/N$ and $2^*_{\alpha,\mu} = (2N - 2\alpha - \mu)/(N-2)$. Our main contribution is to find the largest $\lambda^* > 0$ in such way that our main problem admits at least two positive solutions for each $\lambda \in (0, \lambda^*)$. In order to do that we apply the nonlinear Rayleigh quotient together with the Nehari method. Moreover, we prove a Brezis-Lieb type Lemma and a regularity result taking into account our setting due to the potentials $a, b : \mathbb{R}^N \to \mathbb{R}$.

1 Introduction

In the present work, we consider existence and multiplicity of solutions for nonlocal elliptic problems for the Stein-Weiss type problem. More specifically, we shall consider the following nonlocal elliptic problem:

$$\begin{cases} -\Delta u + V(x)u = \lambda a(x)|u|^{q-2}u + \displaystyle\int_{\mathbb{R}^N} \frac{b(y)|u(y)|^p dy}{|x|^\alpha |x-y|^\mu |y|^\alpha} b(x)|u|^{p-2}u, & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{P_λ}$$

where $\lambda > 0, \alpha \in (0, N), N \geq 3, 0 < \mu < N, 0 < \mu + 2\alpha < N$. Furthermore, we assume also that $a \in L^r(\mathbb{R}^N), a > 0$ in $\mathbb{R}^N$ and $b \in L^t(\mathbb{R}^N), b > 0$ in $\mathbb{R}^N$ for some $r, t > 1$. Here we assume also that $1 \leq q < 2$ and $2_{\alpha,\mu} < p < 2^*_{\alpha,\mu}$ where $2_{\alpha,\mu} = (2N - 2\alpha - \mu)/N$ and $2^*_{\alpha,\mu} = (2N - 2\alpha - \mu)/(N-2)$. It is important to stress that several works have been done in the last years considering the Stein-Weiss problem with different kind of nonlinearities, see [1].

2 Main Results

Throughout this work, we shall assume the following hypotheses:

(H_1) There exists $V_0, V_\infty > 0$ such that $0 < V_0 \leq V(x) \leq V_\infty, x \in \mathbb{R}^N$. Assume also that $\lambda > 0$, $\alpha \in (0, N)$, $N \geq 3$, $0 < 2\alpha + \mu < N$.

(H_2) Suppose that $a > 0$ and $b > 0$ in $\mathbb{R}^N$. Assume also that $1 \leq q < 2$, $2_{\alpha,\mu} < p < 2^*_{\alpha,\mu}$ where

$$2_{\alpha,\mu} = \frac{2N - 2\alpha - \mu}{N} \quad \text{and} \quad 2^*_{\alpha,\mu} = \frac{2N - 2\alpha - \mu}{N-2}.$$

(H_3) It holds $a \in L^r(\mathbb{R}^N),\ b \in L^{st}(\mathbb{R}^N)$ where $2^* = 2N/(N-2)$ and

$$r = \frac{2^*}{2^* - q} \text{ and } ts = \frac{2N}{2N - 2\alpha - \mu - p(N-2)}.$$

(H_4) It holds $a \in L^{N/2}(\mathbb{R}^N),\ \ b \in L^\beta(\mathbb{R}^N) \cap L^\gamma(\mathbb{R}^N)$. Furthermore, we assume that

$$\beta > \max\left\{\frac{N}{2-(N-2)(p-1)-(2\alpha+\mu)+N}, \frac{N}{2-(N-2)(p-1)}\right\}$$

and

$$\gamma = \frac{N\beta}{\beta[2-(N-2)(p-1)+(N-2\alpha-\mu)]-N}.$$

Now, we mention that the energy functional $J_\lambda : H^1(\mathbb{R}^N) \longrightarrow \mathbb{R}$ is associated to the Problem (P_λ) which is defined by

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\mathbb{R}^N} a(x)|u(x)|^q dx - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{b(y)|u(y)|^p b(x)|u(x)|^p}{|x|^\alpha |x-y|^\mu |y|^\alpha} dxdy. \tag{1}$$

Furthermore, we shall consider the Nehari manifold as follows:

$$\mathcal{N}_\lambda = \left\{u \in H^1(\mathbb{R}^N) \setminus \{0\}, J'_\lambda(u)(u) = 0\right\}. \tag{2}$$

Now, we are stay in position to state our main result as follows:

Theorem 2.1. *Suppose $(H_1)-(H_4)$. Then we obtain that $0 < \lambda_* < \lambda^* < \infty$. Furthermore, the Problem (P_λ) admits at least two positive solutions $u_\lambda, v_\lambda \in H^1(\mathbb{R}^N)$ for each $\lambda \in (0, \lambda^*)$. Moreover, the solutions u_λ, v_λ satisfy the following assertions:*

a) $J''_\lambda(u_\lambda)(u_\lambda, u_\lambda) > 0$ *and* $J''_\lambda(v_\lambda)(v_\lambda, v_\lambda) < 0$.

b) The function u_λ is a ground state solution and $J_\lambda(u_\lambda) < 0$.

c) The function v_λ is a bound state solution which has the following properties:

  i) For each $\lambda \in (0, \lambda_*)$, we obtain that $J_\lambda(v_\lambda) > 0$.

  ii) For $\lambda = \lambda_*$ it holds $J_\lambda(v_\lambda) = 0$.

  iii) For each $\lambda \in (\lambda_*, \lambda^*)$ there holds $J_\lambda(v_\lambda) < 0$.

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THE METHOD OF THE NEHARI MANIFOLD ON CONES

D. S. PEREIRA[1], J. R. SANTOS JR.[2], & J. F. F. SILVA[3]

[1]Universidade Federal de Campina Grande, UAME, PB, Brazil, denilsonsp@mat.ufcg.edu.br
[2]Universidade Federal do Pará, ICEN, PA, Brazil, joaojunior@ufpa.br
[3]Universidade Federal do Pará, PDM, PA, Brazil, joaofelipe.ce@gmail.com

Abstract

In this work we establish an abstract method that allows us to study the existence of a ground state solution for some classes of elliptic problems with continuous nonlinearities. Such solutions are found via minimization on the Nehari manifold which, in the abstract situation, is contained in a suitable open cone.

1 Introduction

In [2], among other things, A. Szulkin and T. Weth study a series of elliptic partial differential problems involving continuous (and possibly non-differentiable) nonlinearities whose primitives satisfy the so called superquadratic condition at infinity. In order to apply the Nehari method to find ground-state and other kind of solutions, the authors introduce an interesting approach to overcome the lack of a C^1 structure for the Nehari set. In fact, under suitable conditions on the functional $I : E \to \mathbb{R}$ and on the Banach space E, is proved the existence of a homeomorphism between the Nehari set $\mathcal{N}$ associated to I and the unit sphere $\mathcal{S}$ of E. Such a homeomorphism allows them to define an auxiliary functional $\Psi : \mathcal{S} \to \mathbb{R}$ with the convenient property that the existence of critical points for Ψ implies in the existence of critical points of I. Due to conditions imposed on the Banach space E, the unit sphere $\mathcal{S}$ is a C^1 manifold and, for this reason, the task of looking for critical points of Ψ is a more treatable problem.

Still in [2], it is revisited a classical paper of Benci and Cerami (see [1]) which relate the number of solutions of a certain elliptic PDE with the topology of the domain where the equation is considered. The authors are able to improve the main results in [1] by assuming weaker conditions on the nonlinearity. However, since the idea is to obtain positive solutions, the Nehari set in this case is no longer homeomorphic to the hole unit sphere $\mathcal{S}$, but to an open subset $\mathcal{S}^+$ of $\mathcal{S}$. As observed in [2], this new situation brings together some technical difficulties, as for instance the fact that $\mathcal{S}^+$ has a nonempty boundary in $\mathcal{S}$ and, therefore, the behaviour of minimizing sequences need to be carefully controlled near the boundary.

In this work, we generalize the ideas in [2] to find positive solutions of the Benci-Cerami problem for an abstract situation where the same sort of technical difficulty is occurring. In the general context, we establish an abstract method that allows us to study the existence of solutions for some classes of elliptic problems with continuous nonlinearities. The referred method is used to find ground state solutions to a sort of elliptic problems under different mathematical contexts.

2 Main Results

2.1 An integro-differential problem: Perturbation of the logistic function

In this subsection, we are interested in studying the existence of a ground state solution for the following problem

$$\begin{cases} -\Delta u + a\displaystyle\int_\Omega u\, dx = \lambda u - u^2 + g(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P_2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $1 \leq N < 6$, $a > 0$, $\lambda < \lambda_1(a)$ and $g \in C(\mathbb{R})$ satisfying:

(g_1) $\displaystyle\lim_{s\to 0}\frac{g(s)}{s} = 0$ and $\displaystyle\lim_{|s|\to+\infty}\frac{g(s)}{s} = g_\infty < \lambda_1(a)$;

(g_2) The map $s \mapsto g(s)/|s|$ is nondecreasing;

Theorem 2.1. *Let $a \in (0, (\lambda_2 - \lambda_1)/|\Omega|)$, $\lambda < \lambda_1(a)$ and suppose $(g_1) - (g_2)$ occur. Then, problem (P_2) admits a ground state solution that changes sign or is negative.*

2.2 An integro-differential problem: Superquadratic nonlinearity on a subdomain

In this subsection, we are interested in studying the existence of a ground state solution for the following problem

$$\begin{cases} -\Delta u + a\displaystyle\int_\Omega u\, dx = \lambda u + (1 - \mathcal{X}_{\Omega_0}(x))g(u) + \mathcal{X}_{\Omega_0}(x)f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P_3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $a > 0$, $\lambda < \lambda_1(a)$, $\Omega_0 \subset\subset \Omega$ is an open set and $f, g \in C(\mathbb{R})$ satisfies:

(g_1) $g = 0$, in $(-\infty, 0)$, $\displaystyle\lim_{s\to 0}\frac{g(s)}{s} = 0$ and $\displaystyle\lim_{|s|\to+\infty}\frac{g(s)}{s} = g_\infty < \lambda_1(a)$;

(g_2) $s \mapsto g(s)/|s|$ is a nondecreasing function;

(f_1) $f = 0$, in $(-\infty, 0)$, $\displaystyle\lim_{s\to 0}\frac{f(s)}{s} = f_0 < \lambda_1(a)$ and $\displaystyle\lim_{|s|\to+\infty}\frac{F(s)}{s^2} = +\infty$, $F(s) := \displaystyle\int_0^s f(\tau)\, d\tau$;

(f_2) $s \mapsto f(s)/|s|$ is increasing in $(0, +\infty)$;

Theorem 2.2. *Let $a \in (0, (\lambda_2 - \lambda_1)/|\Omega|)$, $\lambda < \lambda_1(a)$ and suppose $(g_1) - (g_2)$ and $(f_1) - (f_2)$ occur. Then, problem (P_3) admits a non-trivial ground state solution.*

Other applications to problems involving the p-laplacian and Kirchhoff operators are also provided.

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EXISTENCE, NON-EXISTENCE AND DEGENERACY OF SOLUTIONS $P-$LAPLACE PROBLEMS INVOLVING HARDY POTENTIALS AS $P \to 1^+$

J. C. ORTIZ CHATA & F. PETITTA

[1]Universidade Federal de São Carlos, UFSCar, Brazil, juanchata@ufscar.br

[2]Sapienza Università di Roma, Italy, francesco.petitta@uniroma1.it

Abstract

In this work we analyze the asymptotic behaviour as $p \to 1^+$ of solutions u_p to

$$\begin{cases} -\Delta_p u_p &= \frac{\lambda}{|x|^p}|u_p|^{p-2}u_p + f \qquad \text{in } \Omega, \\ u_p &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary, $\lambda \in \mathbb{R}^+$, and f is a nonnegative datum in $L^N(\Omega)$. Under sharp smallness assumptions on the data λ and f, we estimate the family $(u_p)_{p>1}$ uniformly in $BV(\Omega)$ and then we let $p \to 1^+$ in order to completely characterize its limit u. As a consequence of this limit procedure, we prove that u suitably solves the homogeneous Dirichlet problem

$$\begin{cases} -\Delta_1 u &= \frac{\lambda}{|x|}\mathrm{Sgn}(u) + f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $\Delta_1 u = \mathrm{div}\left(\frac{Du}{|Du|}\right)$ is the 1-Laplace operator. The main assumptions are further discussed through explicit examples in order to show their optimality.

1 Introduction

Consider the problem

$$\begin{cases} -\Delta_1 u &= \frac{\lambda}{|x|}\frac{u}{|u|} + f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_1 u = \mathrm{div}\,(\frac{Du}{|Du|})$ is the $1-$Laplacian operator, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an open set with bounded Lipschitz boundary containing the origin, $0 < \lambda < N-1$, and f belongs to $L^N(\Omega)$ satisfying the following assumption

$$\|f\|_{L^N(\Omega)} S_N + \frac{\lambda}{N-1} \leq 1. \tag{2}$$

The standard space to study the problem above is the bounded variation functions space BV and since the quotient $\frac{Du}{|Du|}$ is not defined when $Du = 0$ nor is it meaningful since Du is Radon vector measure, it is replaced by a suitable vector field through the Anzellotti theory, which, in turn, allows to employ the concept of solution introduced in [1] to give a definition of solution to (1).

Problems involving the $1-$Laplacian operator are applied, for instance, in image restoration models [6], torsional problems [3], magnetic resonance electrical impedance tomography [4].

The case $\lambda = 0$ and $f = 1$ in (1) is studied in [3]. The author studied solutions for problem above by means of solutions u_p to the problem

$$\begin{cases} -\Delta_p u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

as $p \to 1^+$, under the smallness condition of the domain. Already in [2], the authors studied this issue under the condition smallness on data f in the Lebesgue space $L^N(\Omega)$ or in the Lorentz space $L^{N,\infty}(\Omega)$.

The main objective of this work is to provide a complete and optimal description of problem (1) under the assumption (2) by analyzing the asymptotic behaviour of the solutions u_p of the problem

$$\begin{cases} -\Delta_p u = \frac{\lambda}{|x|^p}|u|^{p-2}u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

as $p \to 1$. In order to do so, the family $(u_p)_{p>1}$ is estimated uniformly in p and so, the convergence of such family is studied. It is worth noting that for this purpose, due to the presence of zero order term in this problem, truncation arguments and a relation of the parameter λ with the best constant of Hardy inequality are required.

2 Main results

Our main results are as follows.

Theorem 2.1. *Let $0 < \lambda < N-1$. Let $f \in L^N(\Omega)$ satisfy (2). Then there exist $u \in BV(\Omega)$, $\mathbf{z} \in L^\infty(\Omega, \mathbb{R}^N)$ with $\|\mathbf{z}\|_\infty \le 1$ and $s(x) \in Sgn(u(x))$ a.e. $x \in \Omega$ such that*

(1) $-div\,\mathbf{z} = \frac{\lambda}{|x|}s(x) + f \quad$ *in* $\mathcal{D}'(\Omega)$;

(2) $(\mathbf{z}, DT_k(u)) = |DT_k(u)| \quad$ *as measures on* Ω, *for all* $k > 0$;

(3) $[\mathbf{z}, \nu](x) \in Sgn(-u(x)) \quad \mathcal{H}^{N-1}-$ *a.e.* $x \in \partial\Omega$,

where $Sgn(s) = \frac{s}{|s|}$ if $s \neq 0$, $Sgn(0) = [-1, 1]$ and $T_k(s) = s$ if $|s| \le k$, $T_k(s) = k\frac{s}{|s|}$ if $|s| > k$.

Proof Use u_p as test function in (3) to uniformly estimate on p to the family $(u_p)_{p>1}$ and so, letting $p \to 1^+$, by truncation arguments, the limit of this family satisfies conditions (1) – (3).

Theorem 2.2. *If assumption (2) is strictly less than 1,*

$$u_p \to 0 \qquad a.e. \text{ on } \quad \Omega \quad as \quad p \to 1^+$$

and if this assumption is equal to 1, there exist $u \in BV(\Omega)$ such that

$$u_p \to u \qquad in \quad L^s(\Omega),\ s \in [1, 1^*) \quad as \quad p \to 1^+.$$

Proof Use again u_p as teste function in (3) to find uniformly estimate on p to the family $(u_p)_{p>1}$ and so, considering the two cases of the assumption (2), we analyze the asymptotic behaviour of this family as $p \to 1^+$.

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THE EFFECT OF A PERTURBATION ON BREZIS NIRENBERG'S PROBLEM

LUIZ FERNANDO DE OLIVEIRA FARIA[1], JEFERSON CAMILO SILVA[1] & P. UBILLA[2]

[1]Dep. Mathematics, Federal University of Juiz de Fora, MG, Brazil, luiz.faria@ufjf.br, jefersonbs2009@gmail.com
[2]Dep. Matemáticas y C. C., Universidad de Santiago de Chile, Santiago, Chile, pubilla.ubilla@usach.cl

Abstract

In this talk, we consider some critical Brézis-Nirenberg problems in dimension $N \geq 3$ that do not have a solution. We prove that a supercritical perturbation can lead to the existence of a positive solution. More precisely, we consider the equation:

$$\begin{cases} -\Delta u &= \lambda u^{q-1} + u^{2^*+r^\alpha-1} & \text{in} \quad B, \\ u &> 0 & \text{in} \quad B, \\ u &= 0 & \text{on} \quad \partial B, \end{cases}$$

where $B \subset \mathbb{R}^N$ is a unit ball centered at the origin, $N \geq 3$, $r = |x|$, $\alpha \in (0, \min\{N/2, N-2\})$, λ is a fixed real parameter and $q \in [2, 2^*)$. This class of problems can be interpreted as a perturbation of the classical Brézis-Nirenberg problem by the term r^α at the exponent, making the problem supercritical when $r \in (0,1)$. More specifically, we study the effect of this supercritical perturbation on the existence of solutions. In particular, when $N = 3$, an interesting and unexpected phenomenon occurs. We obtain the existence of solutions for λ in a range where the Brézis-Nirenberg problem has no solution.

1 Introduction

In 1983, Brézis and Nirenberg in [1] studied the following problem:

$$\begin{cases} -\Delta u &= \lambda u^{q-1} + u^{2^*-1} & \text{in} \quad \Omega, \\ u &> 0 & \text{in} \quad \Omega, \\ u &= 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 3$, λ is a fixed real parameter, $q \in [2, 2^*)$ and $2^* = 2N/(N-2)$ is the critical exponent in the sense of Sobolev's embedding.

Brézis and Nirenberg proved the following results:

(a) For $q = 2$ and $N \geq 4$, problem (1) has a solution for every $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$. Moreover, it has no solution if $\lambda \notin (0, \lambda_1)$ and Ω is star-shaped.

(b) When $q = 2$, $N = 3$, and Ω is a ball, problem (1) has a solution if and only if $\lambda \in \left(\frac{\lambda_1}{4}, \lambda_1\right)$.

(c) For $q \in (2, 2^*)$ and $N \geq 4$, problem (1) has a solution for every $\lambda > 0$.

(d) When $N = 3$ and $4 < q < 6$, problem (1) has a solution for every $\lambda > 0$.

(e) When $N = 3$ and $2 < q \leq 4$, problem (1) has a solution only for sufficiently large values of λ.

Recently, do Ó, Ruf and Ubilla in [2] studied the following problem:

$$\begin{cases} -\Delta u &= u^{2^*+r^\alpha-1} & \text{in} \quad B, \\ u &> 0 & \text{in} \quad B, \\ u &= 0 & \text{on} \quad \partial B, \end{cases} \tag{2}$$

where $B \subset \mathbb{R}^N$ is the unit ball centered at the origin, $N \geq 3$, $r = |x|$ and $\alpha \in (0, \min\{N/2, N-2\})$.

The authors demonstrated that problem (2) has a radial solution, which is surprising because it corresponds to a supercritical perturbation of the equation $-\Delta u = u^{2^*-1}$, which has no solution due to the known Pohozaev identity. In this same line of reasoning, in the context of the situation of item (b), we studied the effect of a supercritical perturbation for the case of non-existence $\lambda \in (0, \frac{\lambda_1}{4}]$, which also generated the existence of a positive solution. We will also have the same conclusion for situation (e), in which, due to the supercritical perturbation, we will obtain a solution for all positive λ and not just for sufficiently large λ. Motivated by the results of [1] and [2], we studied this problem in a more general context, more precisely, let us consider the following problem:

$$\begin{cases} -\Delta u &= \lambda u^{q-1} + u^{2^*+r^\alpha-1} & \text{in} \quad B, \\ u &> 0 & \text{in} \quad B, \\ u &= 0 & \text{on} \quad \partial B, \end{cases} \tag{3}$$

where $B \subset \mathbb{R}^N$ is a unit ball centered at the origin, $N \geq 3$, $r = |x|$ and $\alpha \in (0, \min\{N/2, N-2\})$ and λ is a fixed real parameter and $q \in [2, 2^*]$.

We will now present the main result.

Theorem 1.1. *If $q = 2$, $\lambda \in [0, \lambda_1)$ and $N \geq 3$, then the problem (3) has a radial weak solution. If $q \in (2, 2^*]$, problem (3) has a radial weak solution for every $\lambda \geq 0$ and $N \geq 3$.*

We also consider some perturbations of Problem (1) that become superlinear on the ball and subcritical for $r \in (0, \delta)$, for some small δ. However, it can be supercritical away from $r = 0$, as in the following equation:

$$\begin{cases} -\Delta u &= \lambda u^{q-1} + u^{2^*+f(r)-1} & \text{in} \quad B, \\ u &> 0 & \text{in} \quad B, \\ u &= 0 & \text{on} \quad \partial B, \end{cases} \tag{4}$$

where $B \subset \mathbb{R}^N$ is a unit ball centered at the origin, $N \geq 3$, $r = |x|$, λ is a fixed real parameter, $q \in [2, 2^*)$ and $f\colon [0,1) \to \mathbb{R}$ is a continuous function satisfying:

(f) $\quad f(0) < 0$ and $\inf\limits_{r \in [0,1)} (2^* + f(r)) > 2$.

The next result involves the assumption (f):

Theorem 1.2. *Let $q \in [2, 2^*)$, $N \geq 3$ and $f\colon [0,1) \to \mathbb{R}$ a continuous function satisfying condition (f). Then the problem (4) has a radial weak solution in the following cases:*

(i) *$q = 2$ and $\lambda \in [0, \lambda_1)$.*

(ii) *$q \in (2, 2^*)$ and $\lambda \geq 0$.*

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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A CLASS OF DIRAC EQUATIONS

C. O. ALVES[1], R. N. DE LIMA[1], & A. B. NÓBREGA[1]

[1] Universidade Federal de Campina Grande, Campina Grande - PB, Brazil,
coalves@mat.ufcg.edu, romildo@mat.ufcg.edu.br, alannio@mat.ufcg.edu.br

Abstract

In this paper we study the existence and multiplicity of solutions for the following class of nonlinear Dirac equations

$$(P) \qquad -i\varepsilon \sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u + V(x)u = f(|u|)u, \quad \text{in} \quad \mathbb{R}^3,$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ and $f : [0, +\infty) \to \mathbb{R}$ are continuous functions. It is proved that the number of solutions is at least the number of global minimum points of V when ε is small enough.

1 Introduction

This paper concerns the existence and multiplicity of solutions for the following class of nonlinear Dirac equations

$$(P) \qquad -i\varepsilon \sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u + V(x)u = f(|u|)u, \quad \text{in} \quad \mathbb{R}^3,$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions. Here, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\partial_k = \frac{\partial}{\partial x_k}$, $a > 0$ is a real constant, $\alpha_1, \alpha_2, \alpha_3$ and β are 4×4 complex matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Related to the potential V we assume the following hypothesis:
(V_1) $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that

$$\lim_{|x| \to \infty} V(x) = V_\infty > V_0 = \min_{x \in \mathbb{R}^N} V(x),$$

with $-a < V_0 \leq V(x) \leq V_\infty < a$, for all $x \in \mathbb{R}^3$.
(V_2) There exist l points $z_1, z_2, \cdots, z_l \in \mathbb{R}^3$ with $z_1 = 0$ such that

$$V(z_i) = V_0, \text{ for } 1 \leq i \leq l.$$

Hereafter, the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:
(f_1) $f(0) = 0$, $f \in C^1((0, \infty), \mathbb{R})$, $f'(t) > 0$ for $t > 0$, and there exist $p \in (2, 3)$ and $c_1 > 0$ such that

$$0 \leq f(t) \leq c_1(1 + t^{p-2}), \quad \forall t \geq 0.$$

(f_2) There exists $\theta > 2$ such that

$$0 < \theta F(|t|) \leq f(|t|)|t|^2, \quad \forall t \neq 0,$$

where $F(|t|) = \int_0^t f(|s|)sds$.

The present work has been motivated by results found in [1], [2] and [3]. In [1], Ding showed the existence and concentration of solution for the following class of nonlinear Dirac equation

$$-i\varepsilon \sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u = P(x)|u|^{p-2}u, \quad \text{in} \quad \mathbb{R}^3, \tag{1}$$

for $p \in (2, 3)$, where the concentration phenomena holds around the maximum points of P. In [2], Ding and Liu established the existence and concentration of solution for problem (P) when $f(t) = |t|^{p-2}t$ and ε is small enough around the minimum point of the potential V, which satisfies the following condition

$$0 < V_0 = \min_{x \in \mathbb{R}^3} V(x) < \liminf_{|x| \to +\infty} V(x). \tag{2}$$

Our main goal of the present paper is to complement the results found in [2] in the following sense: We intend to prove that the number of global minimum points of V is directly related with the numbers of solutions when ε is small, more precisely, if V has l local minimum as in assumption (V_1), then problem (P) has at least l nontrivial solutions when ε is small enough.

2 Main Result

Our main result is the following:

Theorem 2.1. *Assume $(V_1) - (V_2)$ and $(f_1) - (f_2)$. Then, there is $\varepsilon_0 > 0$ such that (P) has at least l nontrivial solutions for all $\varepsilon \in (0, \varepsilon_0)$.*

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NONLOCAL QUAISLINEAR ELLIPTIC PROBLEMS ON IN BOUNDED DOMAINS

T. R. CAVALCANTE[1] & E. D. SILVA[2]

[1]Universidade Federal do Tocantins, TO, Brazil, thiago.cavalcante@mail.uft.edu.br
[2]Universidade Federal de Goiás, GO, Brazil, edcarlos@ufg.br

Abstract

In this we establish existence of solutions for nonlocal elliptic problems driven by the fractional $(p,q)-$ Laplacian. More specifically, we shall consider the following nonlocal elliptic problem :

$$\begin{cases} (-\Delta)_p^{s_1} u - \mu(-\Delta)_q^{s_2} = \lambda |u|^{r-2}u \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P_μ}$$

where$N > s_1 p$, $N > s_2 q$, $s_1 > s_2$ and $r > p > q$. The main feature is to find sharp parameters $\lambda > 0$ and $\mu > 0$ where the Nehari method can be applied finding the largest positive number $\mu^* > 0$ such that our main problem admits at least two distinct solutions for each $\mu \in (0, \mu^*)$. A crucial part of this work is the fact that we consider the term $-\mu(-\Delta)_q^{s_2}$ in the problem (P_μ)

1 Introduction

In the present work we shall consider nonlocal elliptic problems driven by the fractional $(p,q)-$ Laplacian defined in bounded domanin. Namely, we shall consider the following nonlocal elliptic problem

$$\begin{cases} (-\Delta)_p^{s_1} u - \mu(-\Delta)_q^{s_2} = \lambda |u|^{r-2}u \text{ in } \Omega \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{P_μ}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N > s_1 p$, $N > s_2 q$, $s_1 > s_2$ and $r > p > q$.

In order to do that we employ the nonlinear Rayleigh quotient together a fine analysis on the fibering maps associated to the energy functional. It is important to mention also that for each parameters $\lambda > 0$ and $\mu > 0$ there exist degenerate points in the Nehari set which give serious difficulties.

2 Main Results

The working space is defined by $X = \{u \in W^{s,p}(\mathbb{R}^N);\ u = 0 \text{ in} \, \mathbb{R}^N \setminus \Omega\}$ and the energy functional $J : X \to \mathbb{R}$ associated to Problem (P_μ) is given by

$$J_{\lambda,\mu}(u) = \frac{1}{p}[u]_p^p - \frac{\mu}{q}[u]_q^q - \frac{\lambda}{r}\|u\|_r^r,$$

where

$$[u]_p^p := [u]_{s,p}^p = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, dxdy,\ u \in X.$$

In this work, we study the fibers maps of two functionals based on the parameter μ. The first defined for the case where $J'_{\lambda,\mu}(u) = 0$ and the second, considering μ, for which $J(u) = 0$. In short, we consider the functionals $R_n, R_e : X \setminus \{0\} \to \mathbb{R}$ associated with the parameter $\mu > 0$ in the following form

$$R_n(u) = \frac{[u]_p^p - \lambda\|u\|_r^r}{[u]_q^q} \quad \text{and} \quad R_e(u) = \frac{\frac{q}{p}[u]_p^p - \lambda\frac{q}{r}\|u\|_r^r}{[u]_q^q}, u \in X \setminus \{0\},$$

the nonlinear Rayleigh quotients. In association with these, based on work [1], we define the following coefficients

$$\mu^* := \inf_{u\in X\setminus\{0\}} \inf_{t>0} R_n(tu) \qquad \text{and} \qquad \mu^{**} := \inf_{u\in X\setminus\{0\}} \inf_{t>0} R_e(tu). \tag{1}$$

The subset of X, in which the $J_{\lambda,\mu}$ function will be minimized, well known and studied in recent years for Nehari is

$$\mathcal{N}_{\lambda,\mu} = \{u \in X,\, u \neq 0 \;:\; \langle J'_{\lambda,\mu}(u), u\rangle = 0\}.$$

Under these conditions, by using the same ideas considered in [2], we shall split the Nehari manifold $\mathcal{N}_{\lambda,\mu}$ into three disjoint subsets in the following way:

$$\begin{aligned}
\mathcal{N}^+_{\lambda,\mu} &= \{u \in \mathcal{N}_{\lambda,\mu} : J''_{\lambda,\mu}(u)(u,u) > 0\},\\
\mathcal{N}^-_{\lambda,\mu} &= \{u \in \mathcal{N}_{\lambda,\mu} : J''_{\lambda,\mu}(u)(u,u) < 0\},\\
\mathcal{N}^0_{\lambda,\mu} &= \{u \in \mathcal{N}_{\lambda,\mu} : J''_{\lambda,\mu}(u)(u,u) = 0\}.
\end{aligned}$$

We shall state our first main result as follows:

Theorem 2.1. *Suppose $\mu \in (0,\mu^*)$, where μ^* follows from (1). Then there are two solutions $u_1, u_2 \in X \setminus \{0\}$ that satisfy the following statements:*

i) $J''_{\lambda,\mu}(u_1,u_1) < 0$, *that is,* $u_1 \in \mathcal{N}^-_{\lambda,\mu}$;

ii) $J''_{\lambda,\mu}(u_2,u_2) > 0$, *that is,* $u_2 \in \mathcal{N}^+_{\lambda,\mu}$;

iii) $J_{\lambda,\mu}(u_2) < 0$, *for all* $\mu \in (0,\mu^*)$.

Moreover, the weak solution $u_2 \in X$ satisfies the following assertions:

a) For each $0 < \mu < \mu^{**}$, *we obtain* $J_{\lambda,\mu}(u_2) > 0$;

b) For $\mu = \mu^{**}$ *it follows that* $J_{\lambda,\mu}(u_2) = 0$

c) For each $\mu^{**} < \mu < \mu^*$ *we obtain also that* $J_{\lambda,\mu}(u_2) < 0$

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ON A NONLINEAR PROBLEM FOR $P(U)-$LAPLACIAN-LIKE OPERATORS WITH NONLINEAR GRADIENT TERM

WILLY BARAHONA M.[1], GABRIEL RODRÍGUEZ V.[1], GABRIEL RODRÍGUEZ V.[1], ROCÍO DE LA CRUZ M.[1] & JESÚS LUQUE R.[1]

[1]Universidad Nacional Mayor de San Marcos, GI-EDOACBI. FCM, Lima - Perú,
wbarahonam@unmsm.edu.pe, cleugenio@yahoo.com, grodriguezv@unmsm.edu.pe,
rdelacruzm@unmsm.edu.pe, jluquer@unmsm.edu.pe

Abstract

The purpose of this article is to obtain weak solutions for the nonlinear problems for $p(u)-$Laplacian-like operators, originated from a capillary phenomena, with a nonlinearity which depends on the gradient. First, we solve a associated boundary-value local problem are given by using a singular perturbation technique and then we use the Schauder fixed-point theorem for obtain our result, in the framework of variable exponent Sobolev spaces.

1 Introduction

The main objective of this work is to look into the existence of weak solutions for the following local $p(u)$-Laplacian problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p(u)-2}\nabla u + \frac{|\nabla u|^{2p(u)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(u)}}}\right) &= f + g(u)|\nabla u|^{p(u)-1} \qquad \text{in } \Omega, \\ u = 0 \qquad\qquad & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and $N \geq 2$, $p \in C(\overline{\Omega})$ for any $x \in \overline{\Omega}$; f is a given function and $g : \mathbb{R} \to \mathbb{R}$ is a bounded and continuous function that belongs to $L^1(\mathbb{R})$. The study of differential and partial differential equations with variable exponent has been received considerable attention in the last two decades, recently it was extended to the case when the exponent depend both on the space variable x and on the unknown solution u (see [1, 5]). Thus, the problem becomes local and more complicated. Problems of the type (1) can be presented as model for many physical applications, for instance mathematical image processing and computer vision (see [2, **?**]). As far as we are aware, the authors have studied problems only with $p(x)$-Laplacian-like operators, in the context of the study of capillarity phenomena (see [3, 4] and references therein).

2 Notations and Main Results

We need some theorems on $W^{1,p(x)}(\Omega)$ which we call a variable exponent Sobolev space.
Let

$$p : \mathbb{R} \to [1, +\infty[\tag{2}$$

be the nonlinear exponent function. Set $C_+(\overline{\Omega}) = \{p(x) \in C(\overline{\Omega}) : p(x) > 1, \forall x \in \overline{\Omega}\}$; $\quad p^+ = max\{p(x); x \in C(\overline{\Omega})\}$, $p^- = min\{p(x); x \in C(\overline{\Omega})\}$, $M(\Omega) = \{\ u$: u is a real-valued measurable function on $\Omega\ \}$ and

$$L^{p(x)}(\Omega) = \{u \in M(\Omega) : \int_\Omega |u(x)|^{p(x)}dx < \infty\}.$$

We can introduce a norm on $L^{p(x)}(\Omega)$

$$|u|_{p(x)} = inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\}$$

and $(L^{p(x)}(\Omega), |.|_{p(x)})$ becomes a Banach Space. The space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)} \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Of course the norm $\|u\| = |\nabla u|_{L^{p(x)}(\Omega)}$ is an equivalent norm to natural norm in $W_0^{1,p(x)}(\Omega)$. Also, we have the space

$$W_0^{1,p(u)}(\Omega) = \{u \in W_0^{1,1}(\Omega)\} : \int_\Omega |\nabla u|^{p(u)}\, dx < \infty\} \qquad \text{such that } 1 < p(u) < \infty \quad \text{for all } u \in \mathbb{R}$$

in which we will prove the existence of weak solutions for the local Â´problem (1). It is a Banach space for the norm $\|u\|_{W_0^{1,p(u)}(\Omega)}$ when $p(u) \in C(\overline{\Omega})$. Since p is continuous, from a Sobolev embedding we have that $W_0^{1,p(u)}(\Omega)$ is separable and reflexive.

Theorem 2.1. *Assume that*

$$p : \mathbb{R} \to \mathbb{R} \quad \textit{is a Lipschitz-continuous function}$$

and that $n < \alpha \leq p(u) \leq \beta < \infty \quad \forall u \in \mathbb{R}$. *If* $f \in W^{-1,\alpha'}(\Omega)$, *then* (1) *has a weak solution in* $u \in W_0^{1,p(u)}(\Omega)$.

Proof We apply using a singular perturbation technique combined with the theory of Sobolev spaces with exponent variables and the Schauder fixed-point theorem . ■

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AN ASYMPTOTICALLY LINEAR PROBLEM VIA PANKOV MANIFOLDS ON CONES

J. R. SANTOS JR.[1] & M. R. P. ARAÚJO[1]

[1] Universidade Federal do Pará, Belém, PA, Brazil
joaojunior@ufpa.br, marcosraylan@ufpa.br

Abstract

We prove the existence of a signed ground state solution in the mountain pass level for a class of asymptotically linear elliptic problems via Pankov manifold method. The main difficulty is due to the fact that the mountain pass geometry is not satisfied in the hole space. We overcome this technical problem by introducing a new approach involving the Pankov manifold contained in open cones.

1 Introduction

We are interested in the existence of ground state and other nontrivial solutions to the following class of semilinear problems

$$\begin{cases} -\Delta u = \lambda u + f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 1$, $\lambda \geq \lambda_1$ where λ_1 denotes the first eigenvalue of the Laplacian operator and $f \in C(\mathbb{R})$ which is asymptotically linear at the origin and at infinity, that is,

$$\lim_{t\to 0} 2F(t)/t^2 = \alpha > 0, \quad \lim_{|t|\to\infty} 2F(t)/t^2 = \eta > 0 \text{ and } f \text{ satisfies} \tag{f_1}$$

(a) (f_2) $t \mapsto f(t)/|t|$ is increasing

(b) (f_3) $0 < \alpha < \lambda_{m+1} - \lambda_m$ for some $m > 1$ and $\lambda_j < \eta$ with $j > 1$.

It is well known in the literature that asymptotically linear problems can be classified as resonant at infinity (if $\lambda_m = \lambda + \eta$ for some $m \in \mathbb{N}$) or non-resonant at infinity (if $\lambda_m \neq \lambda + \eta$ for all $m \in \mathbb{N}$), where λ_m denotes the m-th eivenvalue of the Laplacian operator.

A. Sulkin and T. Weth [1] studied (1) when $f(t) = f(x,t)$ is continuous and satisfies

(SW_1) $|f(x,t)| \leq c(1 + |u|^{p-1})$ for some $c > 0$ and $p \in (2, 2^*)$;

(SW_2) $f(x,t) = 0(t)$ uniformly in x as $|t| \to 0$;

(SW_3) $F(x,t)/t^2 \to \infty$ uniformly in $x \in \Omega$ as $|t| \to \infty$;

(SW_4) $t \to f(x,t)/|t|$ is strictly increasing on $(\infty, 0)$ and on $(0, \infty)$.

Under these assumptions, the authors, using the Pankov manifold method were able to prove the existence of solution to (1). It is important to highlight that nonlinearity satisfies the superquadraticity condition at infinity (SW_3), which is essential to obtain proposition (2.1). In the case of asymptotically linear nonlinearities, obtaining the proposition (2.1) is not an easy task. Not only that, there are other difficulties to be overcome. In order to cite the main difficulties, we point out that the method consists in proving the existence of a homeomorphism m between Pankov manifold $\mathcal{M}$ and a noncomplete submanifold $\mathcal{S}^+_{\mathcal{A}} = \mathcal{S}^+ \cap \mathcal{A}$ of $H^1_0(\Omega)$ where $\mathcal{S}^+ = \mathcal{S} \cap H^+$ and $\mathcal{A} = \left\{u \in H \backslash F : \|u^+\|^2_\lambda - \|u^-\|^2_\lambda < \eta \int_\Omega u^2 dx\right\}$. This fact brings additional problems. Indeed, it is important to assure that minimizing sequences $\{u_n\}$ for Ψ are not near the boundary of $\mathcal{S}^+_{\mathcal{A}}$.

2 Main Results

The corresponding functional energy (1) is

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx - \int_\Omega F(u)dx$$

it is class C^1 in $H = H_0^1(\Omega)$. Setting $H = H^+ \oplus H^0 \oplus H^-$ and $u = u^+ + u^0 + u^-$, where H^+, H^0, H^- correspond to the positive, zero and negative part of the spectrum of $-\Delta - \lambda$ in H. The number β given by $\beta = \lim_{|t|\to\infty}[(1/2)f(t)t - F(t)]$ is well defined and

$$\beta > \frac{\eta C_2(\tau_{\mathcal{A}}^+)^2(C_1\eta)^{N/2}}{2\lambda_1(\Omega)S(\Omega)^{N/2}}, \tag{β}$$

Proposition 2.1. *Suppose that f satisfies $(f_1)-(f_3)$ then For each $w \in \mathcal{A}$ there exists a unique nontrivial critical point $\widehat{m}(u)$ of $I|_{\widehat{E}(w)}$. Moreover, $\widehat{m}(w)$ is the unique global maximum of $I|_{\widehat{E}(w)}$.*

The map $\widehat{m} : \mathcal{A} \to \mathcal{M}$ that each $u \in \mathcal{A}$ associates the only critical point $\widehat{m}(u)$ of $I|_{\widehat{E}(u)}$ is continuous and $m := \widehat{m}|_{\mathcal{S}_{\mathcal{A}}^+}$ is a homeomorphism between $\mathcal{S}_{\mathcal{A}}^+$ and $\mathcal{M}$. Moreover, $m^{-1}(u) = u^+/\|u^+\|_\lambda$.

Let us consider the maps $\widehat{\Psi} : H^+\backslash\{0\} \to \mathbb{R}$ and $\Psi : \mathcal{S}_{\mathcal{A}}^+ \to \mathbb{R}$, given by $\widehat{\Psi}(u) = I(\widehat{m}(u))$ and $\Psi := \widehat{\Psi}|_{\mathcal{S}_{\mathcal{A}}^+}$. These maps will be very important in our arguments mainly because of their properties, which will be presented in the next result. The proof of such a result can be found in [3].

Proposition 2.2. *Suppose that f satisfies $(f_1)-(f_3)$. Then,*

(i) If $\{u_n\}$ is a $(PS)_c$ sequence for Ψ then $\{m(u_n)\}$ is a $(PS)_c$ sequence for I. If $\{u_n\} \subset \mathcal{M}$ is a bounded $(PS)_c$ sequence for I then $\{m^{-1}(u_n)\}$ is a $(PS)_c$ sequence for Ψ.

(ii) u is a critical point of Ψ if, and only if, $m(u)$ is a nontrivial critical point of I. Moreover,

$$c_{\mathcal{M}} := \inf_{u\in\mathcal{M}} I(u) = \inf_{u\in\mathcal{A}} \max_{\substack{t>0\\ v\in F}} I(tu+v) = \inf_{u\in\mathcal{S}_{\mathcal{A}}^+} \max_{\substack{t>0\\ v\in F}} I(tu+v) = \inf_{u\in\mathcal{S}_{\mathcal{A}}^+} \Psi(u).$$

Proposition 2.3. *Suppose that f satisfies $(f_1)-(f_3)$.*

(i) If $\lambda_{m+k} \neq \lambda + \eta$ for all $k \in \mathbb{N}$, then Ψ satisfies the $(PS)_c$ condition in $\mathcal{S}_{\mathcal{A}}^+$, for all $c \geq c_{\mathcal{M}}$;

(ii) If $\lambda_{m+k} = \lambda + \eta$ for some $k \in \mathbb{N}$ and (β) hold, then Ψ satisfies the $(PS)_c$ condition in $\mathcal{S}_{\mathcal{A}}^+$, for all $c \in [c_{\mathcal{M}}, \inf_{u\in\partial\mathcal{S}_{\mathcal{A}}^+} |[u \neq 0]|)$.

Theorem 2.1. *Suppose that f satisfies $(f_1)-(f_3)$ and (β), then there exists a signed mountain pass ground-state solution for problem (1).*

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CAUCHY SEQUENCES IN VECTOR SPACES EMBEDDED IN QUASILINEAR SPACES: THE CASE OF THE FUZZY NUMBERS

B. LAIATE & D. O. MEDEIROS

[1]Escola de Matemática Aplicada, Fundação Getúlio Vargas, FGV/EMAp, RJ, Brazil
Beatriz.Laiate@fgv.br, debora.medeiros@fgv.br

Abstract

We discuss the structure of Banach spaces of finite dimension embedded in a quasilinear space. In particular, when considering the quasilinear space of fuzzy numbers $\mathbb{R}_{\mathcal{F}}$, vector spaces endowed with mirrored arithmetic operations with the Euclidean space have interesting interpretations. More precisely, the arithmetic operations of $\psi-$cross product and $\psi-$cross division can generate Cauchy sequences of fuzzy numbers under two distinct scenarios: with decreasing and increasing diameter.

1 Introduction

Quasilinear spaces have the feature of quasi-distributive law concerning addition and scalar multiplication. The structure of a quasilinear space is intrinsically related to vector spaces, as proved in [4]. This manuscript focus on the space of fuzzy numbers, denoted by $\mathbb{R}_{\mathcal{F}}$, widely studied in the setting of fuzzy and interval analysis. We henceforth denote a finite set of fuzzy numbers $A_i \in \mathbb{R}_{\mathcal{F}}$ by $\mathcal{A} = \{A_1, \ldots, A_n\}$. We assume $\mathcal{A}$ has the property of *Strong Linear Independence* (SLI, for short) ([1]), so that the arithmetic operations on the so-called $\mathcal{S}(\mathcal{A})-$*linearly correlated fuzzy numbers* are defined from the isomorphism $\psi : \mathbb{R}^n \to \mathcal{S}(\mathcal{A})$ [1, 5].

Since $(\mathcal{S}(\mathcal{A}), \mathcal{D}_\psi)$ is a metric space for any finite SLI set $\mathcal{A} = \{A_1, \ldots, A_n\} \subset \mathbb{R}_{\mathcal{F}}$, we say that the sequence $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{S}(\mathcal{A})$ converges to $B \in \mathcal{S}(\mathcal{A})$ w.r.t. $\mathcal{D}_\psi$ if, for all $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that

$$\mathcal{D}_\psi (B_i, B) < \varepsilon \tag{1}$$

for all $i > N$. We denote $B_i \xrightarrow{\psi} B$ in this case. Similarly, the sequence of $\mathcal{S}(\mathcal{A})-$linearly correlated fuzzy numbers $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{S}(\mathcal{A})$ is a Cauchy sequence in $\mathcal{S}(\mathcal{A})$ if for all $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that the inequality

$$\mathcal{D}_\psi (B_i, B_j) < \varepsilon$$

holds for all $i, j > N$.

Proposition 1.1. *[3] Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be an SLI set of fuzzy numbers such that $\mathbb{R} \subseteq \mathcal{S}(\mathcal{A}) \subseteq \mathbb{R}^{\wedge}_{\mathcal{F}}$, and $A \in \mathcal{S}(\mathcal{A})$ with $[A]_1 = \{a_1\}$. Then for all $n \geq 1$, the equalities follow:*

i) $A^{\odot_\psi^k} = \left(k a_1^{k-1}\right) A -_\psi (k-1) a_1^k$;

ii) $A^{\odot_\psi^{-k}} = \left(\frac{k}{a_1^{k-1}}\right) A_\psi^{-1} -_\psi \frac{k-1}{a_1^k}$ *whenever* $a_1 \neq 0$.

2 Main Results

The next theorem follows immediately from the fact that for a given SLI set $\mathcal{A}$, the spaces $(\mathcal{S}(\mathcal{A}), +_\psi, \cdot_\psi, \|\cdot\|_\psi)$ and $(\mathbb{R}, +, \cdot, \|\cdot\|_\infty)$ are isometric.

Theorem 2.1. *Let $\mathcal{A} \subset \mathbb{R}_\mathcal{F}$ be an SLI set and $\{B_i\}_{i\in\mathbb{N}} \subset \mathcal{S}(\mathcal{A})$ be a sequence given by $B_i = q_{i1}A_1 + \ldots + q_{in}A_n$ for all $i \in \mathbb{N}$. Then, the sequence $\{B_i\}_{i\in\mathbb{N}}$ converges to $B \in \mathcal{S}(\mathcal{A})$ w.r.t. $\mathcal{D}_\psi$ if, and only if, the sequence $(q_{i1}, \ldots, q_{in})$ converges to $(q_1, \ldots, q_n)$ in $\mathbb{R}^n$, that is, the following equivalence holds:*

$$B_i \xrightarrow{\psi} B \Leftrightarrow (q_{i1}, \ldots, q_{in}) \to (q_1, \ldots, q_n), \tag{2}$$

where $B = q_1 A_1 + \ldots + q_n A_n$.

Proof. The proof is immediate. □

The next result establishes a condition to the sequences of power hedges of an $\mathcal{S}(\mathcal{A})$-linearly correlated fuzzy number w.r.t. $\odot_\psi$ to be Cauchy sequences in $\mathcal{S}(\mathcal{A})$.

Proposition 2.1. *Let $\mathcal{A}$ be a finite SLI set satisfying $\mathbb{R} \subseteq \mathcal{S}(\mathcal{A}) \subseteq \mathbb{R}_\mathcal{F}^\wedge$ and $A \in \mathcal{S}(\mathcal{A})$ with $[A]_1 = \{a_1\}$. The following properties hold true:*

i) $\left\{A^{\odot_\psi^i}\right\}_{i\in\mathbb{N}} \subset \mathcal{S}(\mathcal{A})$ *is a Cauchy sequence whenever $a_1 \in (-1, 1)$. Moreover, $A^{\odot_\psi^i} \xrightarrow{\psi} 0$;*

ii) $\left\{A^{\odot_\psi^{-i}}\right\}_{i\in\mathbb{N}} \subset \mathcal{S}(\mathcal{A})$ *is a Cauchy sequence whenever $a_1 \in (-\infty, -1) \cup (1, \infty)$. Moreover, $A^{\odot_\psi^{-i}} \xrightarrow{\psi} 0$,*

where $0 \in \mathbb{R}_\mathcal{F}$ is regarded as a singleton.

Proof. *i)* Let $[A]_1 = \{a_1\}$ with $a_1 \in (-1, 1)$. By Proposition 1.1, we have that for all $j, k \in \mathbb{N}$,

$$\begin{aligned}\mathcal{D}_\psi\left(A^{\odot_\psi^j}, A^{\odot_\psi^k}\right) = \left\|A^{\odot_\psi^j} -_\psi A^{\odot_\psi^k}\right\|_\psi &= \left\|\left(ja_1^{j-1}A -_\psi (j-1)a_1^j\right) -_\psi \left(ka_1^{k-1}A -_\psi (k-1)a_1^k\right)\right\|_\psi \\ &= \left\|\left(ja_1^{j-1} - ka_1^{k-1}\right)A -_\psi \left(ja_1^j - ka_1^k\right)\right\|_\psi.\end{aligned}$$

Suppose wlog that $j \le k$. Then,

$$\begin{aligned}\mathcal{D}_\psi\left(A^{\odot_\psi^j}, A^{\odot_\psi^k}\right) &= \left\|ja_1^{j-1}\left(1 - \frac{k}{j}a_1^{k-j}\right)A -_\psi ja_1^j\left(1 - \frac{k}{j}a_1^{k-j}\right)\right\|_\psi \\ &= \left|ja_1^{j-1}\right|\left|1 - \frac{k}{j}a_1^{k-j}\right|\|A -_\psi a_1\|_\psi \to 0\end{aligned}$$

whenever $j, k \to \infty$. In addition,

$$\begin{aligned}\mathcal{D}_\psi\left(A^{\odot_\psi^i}, 0\right) = \mathcal{D}_\psi\left(\left(ia_1^{i-1}\right)A -_\psi (i-1)a_1^i, 0\right) &= \left\|\left(ia_1^{i-1}\right)A -_\psi (i-1)a_1^i\right\|_\psi \\ &\le \left|ia_1^{i-1}\right|\|A\|_\psi - \left|(i-1)a_1^i\right| \to 0 \text{ when } i \to \infty,\end{aligned}$$

that is, $A^{\odot_\psi^i} \xrightarrow{\psi} 0$.

ii) The proof is analogous to item *i)*.

□

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A SUFFICIENT CONDITION FOR A TWISTED G-SUM OF TWO $G_{\mathcal{TOP}}$-BANACH SPACES TO BE A $G_{\mathcal{TOP}}$-BANACH SPACE

D. A. P. GARCIA

Universidade de São Paulo, IME, SP, Brazil, denis.de.assis.pinto.garcia@gmail.com

Abstract

Let G be a topological group. In this work, we introduce the notion of $G_{\mathcal{T}op}$-Banach space (which is a generalization of Castillo and Ferenczi's concept of G-Banach space) and present a sufficient condition for a twisted G-sum of two $G_{\mathcal{T}op}$-Banach spaces to be a $G_{\mathcal{T}op}$-Banach space. More specifically, we state that, if (G, Z, λ) is a twisted G-sum of two $G_{\mathcal{T}op}$-Banach spaces such that Z is either super-reflexive or reflexive and separable, then (G, Z, λ) is also a $G_{\mathcal{T}op}$-Banach space. [1] [2]

1 Introduction

Let G be a topological group, and let e_G and τ_G be, respectively, its identity element and its topology. The main objective of this work is to introduce the concept of $G_{\mathcal{T}op}$-Banach space (which is a generalization of Castillo and Ferenczi's concept of G-Banach space) and present a sufficient condition for a twisted G-sum of two $G_{\mathcal{T}op}$-Banach spaces to be a $G_{\mathcal{T}op}$-Banach space

2 G-Banach spaces and $G_{\mathcal{T}op}$-Banach spaces

In this section, we recall Castillo and Ferenczi's notion of G-Banach space, introduced in [1], and present the concept of $G_{\mathcal{T}op}$-Banach space. To do so, let us start by making a preliminary definition.

Definition 2.1. *A **bounded left action** of G on a normed space X is a map u from G into the set $\mathcal{B}(X)$ of the bounded linear maps from X into X such that: i) $u(e_G) = \mathrm{id}_X$; ii) for each $(g, h) \in G \times G$, $u(g \cdot h) = u(g) \circ u(h)$; and iii) $u(G)$ is a bounded subset of $\mathcal{B}(X)$. If u is a bounded left action of G on a normed space X such that $u(G)$ is a subgroup of the isometry group of X, we say that $||\cdot||_X$ is u-**invariant**.*

Definition 2.2 (see [1])**.** *A G-**Banach space** is an ordered triple (G, X, u), where X is a Banach space, and u is a bounded left action of G on X.*

Definition 2.3. *We say that a G-Banach space (G, X, u) is a $G_{\mathcal{T}op}$-**Banach space** if u is (τ_G, SOT)-continuous.*

Next, let us recall the notions of G-equivariant map and G-operator.

Definition 2.4. *We say that a G-Banach space (G, X, u) is a $G_{\mathcal{T}op}$-**Banach space** if u is (τ_G, SOT)-continuous.*

Next, let us recall the notions of G-equivariant map and G-operator.

Definition 2.5. *Given G-Banach spaces (G, X, u) and (G, Y, v), we say that a map $T\colon X \to Y$ is G-**equivariant** if, for each $g \in G$, $T \circ u(g) = v(g) \circ T$. If, in addition to being G-equivariant, T is linear and continuous, we say that T is a G-**operator**.*

[1] Advisor: V. Ferenczi, Universidade de São Paulo, IME, USP, SP, Brazil, ferenczi@ime.usp.br
[2] We also present a more technical and slightly more general theorem, from which this result can be proven (see theorem 2.1).

From the notions of G-Banach space and G-operator, we can now define the category of G-Banach spaces.

Definition 2.6. *The* ***category of G-Banach spaces*** *is the category that has G-Banach spaces as objects and G-operators as morphisms.*

3 Twisted G-sums of G-Banach spaces

Now that we have presented the concept of G-Banach space, we can introduce the notion of twisted G-sum of two G-Banach spaces.

Definition 3.1. *Let (G, X, u) and (G, Y, v) be G-Banach spaces. We say that a G-Banach space (G, Z, λ) is a* ***twisted G-sum*** *of (G, X, u) and (G, Y, v) if there are G-operators $\iota\colon X \to Z$ and $q\colon Z \to Y$ such that*

$$0 \longrightarrow (G, X, u) \xrightarrow{\iota} (G, Z, \lambda) \xrightarrow{q} (G, Y, v) \longrightarrow 0 \tag{1}$$

is an exact sequence in the category of G-Banach spaces.

The following proposition follows easily from the previous definition.

Proposition 3.1. *If a twisted G-sum of two G-Banach spaces is a $G_{\mathcal{T}op}$-Banach space, then these spaces are also $G_{\mathcal{T}op}$-Banach spaces.*

In light of this result, it is natural to wonder whether its converse is also true. The answer to this question, however, is negative. This leads us to the main objective of this work, which is, precisely, to exhibit a sufficient conditions for a twisted G-sum of two $G_{\mathcal{T}op}$-Banach spaces to be a $G_{\mathcal{T}op}$-Banach space. The next theorem presents us with such a condition.

Theorem 3.1. *If (G, Z, λ) is a twisted G-sum of two $G_{\mathcal{T}op}$-Banach spaces such that Z is reflexive and admits a $\lambda(G)$-invariant LUR renorming, then (G, Z, λ) is also a $G_{\mathcal{T}op}$-Banach space.*

It can be shown that, if (G, Z, λ) is a G-Banach space such that Z is either super-reflexive (and hence reflexive) or reflexive and separable, then there exists a LUR norm on Z which is $\lambda(G)$-invariant and equivalent to $||\cdot||_Z$. This leads us to the following corollary.

Corollary 3.1. *A twisted G-sum of two $G_{\mathcal{T}op}$-Banach spaces whose underlying Banach space is either super-reflexive or reflexive and separable is necessarily a $G_{\mathcal{T}op}$-Banach space.*

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SPEAR VECTORS IN SPACES OF M-HOMOGENEOUS POLYNOMIALS

E. R. SANTOS

Universidade Federal de Uberlândia, IME, MG, Brazil, elisars@ufu.br

Abstract

In this work, we introduce some findings about spear vectors in spaces of m-homogeneous polynomials, generalizing some valid results in the linear case.

1 Introduction

In 2014, M. A. Ardalani [1] introduced the concept of spear vector. Given a Banach space X, a norm-one element $z \in X$ is called a *spear vector* if $\max_{\lambda\in\mathbb{T}} \|z+\lambda x\| = 1 + \|x\|$ for every $x \in X$. We denote by $\text{Spear}(X)$ the set of all spear vectors of X. For $X = \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we have $\text{Spear}(X) = \mathbb{T} = \{\lambda \in \mathbb{K} : |\lambda| = 1\}$. If X is ℓ_1^2, ℓ_∞^2, ℓ_1^3 or ℓ_∞^3, then it is possible to see that $\text{Spear}(X)$ is the set of all extreme points of the unit ball. Moreover, according to [2], $\text{Spear}(\ell_1) = \{\lambda e_n : |\lambda| = 1, n \in \mathbb{N}\}$, $\text{Spear}(\ell_\infty) = \{(a_n) \in \ell_\infty : |a_n| = 1\}$, and $\text{Spear}(C(K)) = \{f \in C(K) : |f(t)| = 1 \text{ for every } t \in K\}$.

V. Kadets, M. Martín, J. Merí, and A. Pérez presented in [2] a deep study of spear vectors in the space $\mathcal{L}(X, Y)$ of all bounded linear operators from X to Y and introduced the concept of spear set. Given a Banach space X, a subset F of the unit ball B_X is called a *spear set* if $\sup_{z\in F} \max_{\lambda\in\mathbb{T}} \|z + \lambda x\| = 1 + \|x\|$ for every $x \in X$. Note that if $F \subset B_X$ is a spear set, then every subset of B_X containing F is also a spear set. In particular, if a subset F of B_X contains a spear vector, then F is a spear set.

Motivated by [1] and [2], in this work, we investigate spear vectors and spear sets in the Banach space $\mathcal{P}(^m X, Y)$ of all continuous m-homogeneous polynomials from X to Y.

2 Main Results

Given a Banach space X, we denote by S_X the unit sphere of X, by X^* the topological dual of X, and by $\mathcal{P}(^m X)$ the Banach space of all continuous m-homogeneous polynomials from X to $\mathbb{K}$. For a nonempty $A \subset X$, we write $\text{conv}(A)$ and $\text{ext}(A)$, to denote the convex hull of A and the set of extreme points of A, respectively. Moreover, a *slice* of A is a set of the form

$$S(A, x^*, \varepsilon) = \left\{ x \in A : \text{Re } x^*(x) > \sup_{a\in A} \text{Re } x^*(a) - \varepsilon \right\},$$

where $x^* \in X^*$ and $\varepsilon > 0$. If $A \subset X^*$ and the functional x defining the slice is taken in the predual, then $S(A, x, \varepsilon)$ is called a *w^*-slice* of A.

In the spirit of the concepts of numerical index with respect to an operator and polynomial numerical index, we introduce the notion of polynomial numerical index with respect to a polynomial.

Definition 2.1. *Let X, Y be Banach spaces, let $Q \in \mathcal{P}(^m X, Y)$ be a norm-one polynomial, and let $m \in \mathbb{N}$. We define the* ***approximated spatial numerical range of P with respect to Q*** *by*

$$V_Q(P) = \bigcap_{\varepsilon>0} \overline{\{y^*(Px) : y^* \in S_{Y^*}, x \in S_X, \text{Re } y^*(Qx) > 1 - \varepsilon\}},$$

*the **numerical radius of** P with respect to Q by*

$$v_Q = \sup\{|\lambda| : \lambda \in V_Q(P)\},$$

*and the **polynomial numerical index of** (X,Y) **with respect to** Q **of order** m by*

$$n_Q{}^{(m)}(X,Y) = \inf\{v_Q(P) : P \in \mathcal{P}(^m X, Y), \|P\| = 1\}.$$

Inspired by Proposition 3.2 by V. Kadets et al. [2], we present the following characterization of $\text{Spear}(\mathcal{P}(^m X,Y))$.

Proposition 2.1. *Let X, Y be Banach spaces and let $Q \in \mathcal{P}(^m X,Y)$ be a norm-one polynomial. The following assertions are equivalent:*

(i) $Q \in \text{Spear}(\mathcal{P}(^m X,Y))$.

(ii) $|\varphi(Q)| = 1$ *for every* $\varphi \in \text{ext}(B_{\mathcal{P}(^m X,Y)^*})$.

(iii) For every $\mathcal{A}$ such that $B_{\mathcal{P}(^m X,Y)^*} = \overline{\text{conv}}^{w^*}(\mathcal{A})$ and every $\varepsilon > 0$, the slice $S(\mathcal{A}, Q, \varepsilon)$ is norming for $\mathcal{P}(^m X,Y)$.

(iv) For every $\varepsilon > 0$ and $P \in \mathcal{P}(^m X,Y)$,

$$\|P\| = \sup\{|y^*(Px)| : y^* \in S_{Y^*}, x \in S_X, \text{Re } (Qx) > 1 - \varepsilon\}.$$

(v) $n_G{}^{(m)}(X,Y) = 1$.

Next, we introduce the concept of the m-order alternative Daugavet property for a m-homogeneous polynomial.

Definition 2.2. *Let X, Y be Banach spaces. We say that $Q \in \mathcal{P}(^m X,Y)$ has the **m-order alternative Daugavet property** (m-ADP in short), if the norm equality*

$$\max_{\lambda \in \mathbb{T}} \|Q + \lambda P\| = 1 + \|P\| \tag{ADE}$$

holds for every rank-one polynomial $P \in \mathcal{P}(^m X,Y)$.

Finally, generalizing Theorem 3.6 by V. Kadets et al. [2], we present a characterization of m-homogeneous polynomials with the m-ADP. For that, we need one last definition. A *k-polynomial slice of* B_X is a set of the form

$$S(p,\varepsilon) = \{x \in B_X : |p(x)| > 1 - \varepsilon\},$$

where $p \in S_{\mathcal{P}(^m X)}$ and $\varepsilon > 0$.

Theorem 2.1. *Let $Q \in \mathcal{P}(^m X,Y)$ be a norm-one polynomial between two Banach spaces X, Y. The following assertions are equivalent:*

(i) Q has the m-ADP.

(ii) $Q(S)$ is a spear set of Y for every m-polynomial slice S of B_X.

(iii) $Q^*(S^*)$ is a spear set of $\mathcal{P}(^m X)$ for every w^*-slice S^* of B_{Y^*}.

(iv) For every $p \in \mathcal{P}(^m X)$, the set

$$\left\{ y^* \in \text{ext} B_{Y^*} : \max_{\lambda \in \mathbb{T}} \|Q^* y^* + \lambda p\| = 1 + \|p\| \right\}$$

is a dense G_δ set in $(\text{ext} B_{Y^*}, w^*)$.

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A SEARCH FOR CONVERGENCE: SERIES IN NON-LINEAR ENVIRONMENTS

G. RIBEIRO

Universidade Federal da Paraíba, DM, PB, Brazil, geivison.ribeiro@academico.ufpb.br

Abstract

In this talk, we will explore the concept of $[S]$-lineability, presenting some characterizations in the context of F-spaces and discussing the absence of $[S]$-lineability in certain subsets of normed and p-Banach spaces. Additionally, we will highlight some open problems related to this notion.

1 Introduction

The study of linearity in unconventional mathematical contexts has gained prominence in recent years. This interest is partly due to the terms *lineability* and *spaceability*, introduced by V.I. Gurariy and popularized by the work of Aron, Gurariy, and Seoane-Sepúlveda (see [2], see also [7]). These concepts have inspired numerous research efforts to identify linear structures in various mathematical areas. A set A in a vector space X is considered lineable if $A \cup \{0\}$ contains an infinite-dimensional vector subspace. If X is a topological vector space, A is considered spaceable if $A \cup \{0\}$ includes a closed infinite-dimensional vector subspace. Further information on lineability can be found in [2, 3, 5, 6].

In 2004, Gurariy introduced the concept of $[\mathcal{S}]$-lineability (see [4]): Given a Hausdorff topological vector space $X \neq \{0\}$ and a vector subspace $\mathcal{S}$ of $\mathbb{K}^{\mathbb{N}}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), a set A in X is called:

- $[(u_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable in X if, for each sequence $(c_n)_{n=1}^{\infty} \in \mathcal{S}$, the series $\sum_{n=1}^{\infty} c_n u_n$ converges in X to a vector in $A \cup \{0\}$.
- $[\mathcal{S}]$-lineable in X if it is $[(u_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable for some sequence $(u_n)_{n=1}^{\infty}$ of linearly independent elements in X.

As far as we know, $[\mathcal{S}]$-lineability has been developed only in [4].

2 Main Results

Among the results we have obtained, we highlight:

1. In an F**-space** $X \neq \{0\}$, if $\{Y_i\}_{i\in I}$ is a family of nontrivial closed subspaces of X, then the set $X \setminus \bigcup_{i\in I} Y_i$ is **lineable** if and only if it is $[\mathcal{S}]$-lineable for every subspace $\mathcal{S} \neq \{0\}$ of ℓ_∞.
2. For an F**-space** E, if W be a subspace of E that contains a non-minimal closed subspace, then the set $E \setminus W$ is **spaceable** if and only if it is $[(x_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable for some closed subspace $\mathcal{S}$ of ℓ_∞ containing c_0 and some $\mathcal{S}$-independent sequence $(x_n)_{n=1}^{\infty}$ of elements of E.
3. If in a **Hausdorff topological vector space** $X \neq \{0\}$, $(x_n)_{n=1}^{\infty}$ is an ℓ_∞-independent sequence of elements of X, and if a nonempty subset A of X is $[(x_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable for some infinite dimensional subspace $\mathcal{S}$ of ℓ_∞, then A is lineable.
4. In an **infinite dimensional normed space** X (or a p-**Banach space**), if $(x_n)_{n=1}^{\infty}$ is a linearly independent sequence in X such that $\inf_{n\in\mathbb{N}} \|x_n\|_X > 0$, then for any infinite dimensional subspace $\mathcal{S}$ of ℓ_∞ properly containing c_0, there is no subset of X that is $[(x_n)_{n=1}^{\infty}, \mathcal{S}]$-lineable.

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NEW BANACH LATTICES OF HOMOGENEOUS POLYNOMIALS

GERALDO BOTELHO[1], VINÍCIUS C. C. MIRANDA[2] & PILAR RUEDA[3]

[1]Universidade Federal de Uberlândia, IME, MG, Brazil, botelho@ufu.br
[2]Universidad Federal do ABC, CMCC, SP, Brazil, colferaiv@gmail.com
[3]Universidad de Valencia, Spain, pilar.rueda@uv.es

Abstract

In this talk we will present the results contained [1, Sections 2 and 3]. The main purpose is to develop a technique to construct new Banach lattices of homogeneous polynomials. We obtain, in particular, conditions for the linear spans of all positive compact and weakly compact n-homogeneous polynomials between Banach lattices to be Banach lattices with the polynomial regular norm. Banach lattices of almost limited polynomials and of solid p-compact polynomials are also obtained. Most of our results and examples are new even in the linear case $n = 1$.

1 Introduction

Let E and F be Banach lattices and let $n \in \mathbb{N}$ be given. The space $\mathcal{P}(^nE;F)$ of continuous n-homogeneous polynomials from E to F is a Banach space with the usual supremum norm. But, in general, it is not a Banach lattice. The usual technique to construct a Banach lattice of n-homogeneous polynomials from E to F is the following (see, e.g., [3]): A polynomial $P \in \mathcal{P}(^nE;F)$ is said to be *positive* if the symmetric n-linear operator associated to P is positive. The difference of two positive n-homogeneous polynomials is called a regular homogeneous polynomial, and the set of all these polynomials is denoted by $\mathcal{P}^r(^nE,F)$. If F is Dedekind complete, then $\mathcal{P}^r(^nE,F)$ is a Banach lattice with the regular norm

$$\|P_r\| = \||P|\| = \inf\{\|Q\| : Q \in \mathcal{P}^+(^nE;F),\, Q \geq |P|\},$$

where $|P|$ denotes the absolute value of P.

In Banach space theory, several closed subspaces of $\mathcal{P}(^nE;F)$ play an important role, for example, spaces of compact and weakly compact polynomials (see [5]). Moreover, other important classes of polynomials are subspaces $\mathcal{A}$ of $\mathcal{P}(^nE;F)$ endowed with a specific complete norm $\|\cdot\|_{\mathcal{A}}$, for example, spaces of nuclear polynomials (see [5]) and p-compact polynomials (see [1]). As expected, none of these Banach spaces of polynomials are Banach lattices in general. The purpose of this talk is to describe a technique to generated Banach lattices of polynomials belonging to $\mathcal{A}$. More than creating many new examples of Banach lattices of homogeneous polynomials, the results we prove and the examples we provide give, in the linear case $n = 1$, new examples of Banach lattices of linear operators.

2 Main Results

Given a vector subspace $\mathcal{A}$ of $\mathcal{P}(^nE;F)$, we denote by $\mathcal{A}^+$ the class of all positive n-homogeneous polynomials belonging to $\mathcal{A}$. We say that the ordered pair (E,F) satisfies the $\mathcal{A}$-domination property if, for all positive n-homogeneous polynomials $P,Q\colon E \to F$ with $0 \leq P \leq Q \in \mathcal{A}$, it holds $P \in \mathcal{A}$.

Theorem 2.1. *Let E and F be Banach lattices with F Dedekind complete and let $\mathcal{A}$ be a subspace of $\mathcal{P}(^nE;F)$ endowed with a complete norm $\|\cdot\|_{\mathcal{A}}$ satisfying the following conditions:*
(I) $\|P\| \leq \|P\|_{\mathcal{A}}$ *for every $P \in \mathcal{A}$.*

(II) $\|P\|_{\mathcal{A}} \leq \|Q\|_{\mathcal{A}}$ *for all* $P \in \mathcal{A}$ *and* $Q \in \mathcal{A}^+$ *with* $|P(x)| \leq Q(|x|)$ *for every* $x \in E$.
Then

$$\|P\|_{\mathcal{A},r} := \inf\{\|Q\|_{\mathcal{A}} : Q \in \mathcal{A}^+, Q \geq |P|\}$$

defines a complete norm on $\mathcal{A}^r := span\{\mathcal{A}^+\}$ *such that* $\|P\|_{\mathcal{A},r} \geq \|P\|_r$ *for every* $P \in \mathcal{A}^r$. *If, in addition,* (E, F) *satisfies the* $\mathcal{A}$*-domination property, then* $(\mathcal{A}^r, \|\cdot\|_{\mathcal{A},r})$ *is a Banach lattice and* $\|\cdot\|_{\mathcal{A}} \leq \|\cdot\|_{\mathcal{A},r}$. *Moreover, in this case,* $\mathcal{A}^r$ *is an ideal in* $\mathcal{P}^r({}^nE; F)$.

Corollary 2.1. *Let* E *and* F *be Banach lattices with* F *Dedekind complete. If* $\mathcal{A}$ *is a closed subspace of* $\mathcal{P}({}^nE; F)$ *such that* (E, F) *satisfies the* $\mathcal{A}$*-domination property, then*

$$\|P\|_{\mathcal{A},r} := \inf\{\|Q\| : Q \in \mathcal{A}^+, Q \geq |P|\}$$

defines a complete lattice norm on $\mathcal{A}^r = span\{\mathcal{A}^+\}$, *that is,* $(\mathcal{A}^r, \|\cdot\|_{\mathcal{A},r})$ *is a Banach lattice. Moreover,* $\|P\|_{\mathcal{A},r} = \|P\|_r$ *for every* $P \in \mathcal{A}^r$ *and* $\mathcal{A}^r$ *is an ideal in* $\mathcal{P}^r({}^nE; F)$.

By $\mathcal{P}_{\mathcal{K}}$ and $\mathcal{P}_{\mathcal{W}}$ we denote the classes of compact and weakly compact homogeneous polynomials.

Examples. (a) If E is a Banach lattice and F is an atomic Banach lattice with order continuous norm, we obtain from Corollary 2.1 that $(\mathcal{P}^r_{\mathcal{K}}({}^nE; F), \|\cdot\|_{\mathcal{K},r})$ is a Banach lattice such that $\|P\|_{\mathcal{K},r} = \|P\|_r$ for every $P \in \mathcal{P}^r_{\mathcal{K}}({}^nE; F)$.

(b) If E is a Banach lattice and F is a Banach lattice with order continuous norm, we obtain from Corollary 2.1 that $(\mathcal{P}^r_{\mathcal{W}}({}^nE; F), \|\cdot\|_{\mathcal{W},r})$ is a Banach lattice such that $\|P\|_{\mathcal{W},r} = \|P\|_r$ for every $P \in \mathcal{P}^r_{\mathcal{W}}({}^nE; F)$.

(c) An n-homogeneous polynomial $P \colon E \to F$ is said to be *almost limited*, in symbols $P \in \mathcal{P}_{\rm al}({}^nE; F)$, if $P(B_E)$ is an almost limited subset of F, that is, for every disjoint weak* null sequence $(y_n^*)_n \subset F^*$, $\|y_n^* \circ P\| = \sup_{x \in B_E} y_n^*(P(x)) \to 0$. If F is Dedekind complete, we obtain from Corollary 2.1 that $(\mathcal{P}^r_{\rm al}({}^nE; F), \|\cdot\|_{{\rm al},r})$ is a Banach lattice such that $\|P\|_{{\rm al},r} = \|P\|_r$ for every $P \in \mathcal{P}^r_{\rm al}({}^nE; F)$.

(d) An n-homogeneous polynomial $P \colon E \to F$ is said to be *solid p-compact*, $1 < p < \infty$, in symbols $P \in \mathcal{P}_{|\mathcal{K}_p|}({}^nE; F)$, if there exists an absolutely p-summable F-valued sequence $(y_j)_j$ such that $P(B_E) \subseteq$ sol$\{p$-conv$\{(y_j)_j\}\}$. This is a Banach space satisfying conditions (I) and (II) of Theorem 1.1 (hardwork!) with the norm

$$\|P\|_{|\mathcal{K}_p|} = \inf\left\{\|(y_j)_j\|_p : P(B_E) \subseteq \mathrm{sol}\{p\text{-conv}\{(y_j)_j\}\}\right\}.$$

From Theorem 1.1 we obtain that $(\mathcal{P}^r_{|\mathcal{K}_p|}({}^nE; F), \|\cdot\|_{||\mathcal{K}_p|^r})$ is a Banach lattice. For $p = 1$, everything holds if E contains no copy of c_0.

Remark. The linear case of Example (a) above was obtained in [4]. The linear cases of (b), (c) and (d) provide new examples of Banach lattices of linear operators.

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INJECTIVE TYPE NORMS AND INTEGRAL BILINEAR FORMS DEFINED BY SEQUENCE CLASSES

J. R. CAMPOS[1], L. NASCIMENTO[2] & L. F. P. SOUSA[3]

[1]Universidade Federal da Paraíba - Campus IV, DCX, PB, Brazil, jamilson@dcx.ufpb.br
[2]Instituto Federal de Educação, Ciência e Tecnologia do Ceará, CE, Brazil, lucas.carvalho@ifce.edu.br
[3]Universidade Federal da Paraíba, DM, PB, Brazil, lfpinhosousa@gmail.com

Abstract

In this work we define classes of injective norms for tensor products through the abstract environment of sequence classes. Examples and results on these norms will be presented and the dual of the tensor product will be constructed, when equipped with one of these norms. This dual leads us to the definition of a class of integral type bilinear forms.

1 Introduction

The study of the duals $(E\hat{\otimes}_\alpha F)'$ (E, F are Banach spaces and α is a tensor norm) as classes of linear operators or bilinear forms is a fundamental part in the theory of tensor products and establishes its close relationship with the parallel theory of operator ideals. It is well known that since the injective norm on $E \otimes F$ is smaller than the projective norm, every bounded linear functional on $E\hat{\otimes}_\varepsilon F$ is the linearization of a unique bounded bilinear form on $E \times F$. A complete description of the dual space of $E\hat{\otimes}_\varepsilon F$ leads to the definition of Integral Bilinear Forms and these lead to the definition of Integral Operators (see [5, Section 3.4]).

The purpose of this talk is to present the definitions of some classes of injective type norms for tensor products and to achieve this goal we use the abstract environment of Sequence Classes [1]. This environment is a unifying approach to deal with operator ideals defined, or characterized, by the transformation of vector-valued sequences. We give examples and establish results on these norms as well as presenting the construction of the dual of the tensor product, when equipped with one of these norms.

Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ are denoted by E and F. The symbol $E \overset{1}{\hookrightarrow} F$ means that E is a linear subspace of F and $\|x\|_F \leq \|x\|_E$ for every $x \in E$. We denote by E' the topological dual of E, B_E denotes the closed unit ball of E and $\mathcal{L}(E;F)$ the Banach space of bounded linear operators from E to F with the usual operator norm. The basic theory, nomenclature and symbology of the sequence class environment are present in [1] and the additional associated elements in [2, 3, 4]. The other notations and symbols used here are either usual in functional analysis or are also present in the references already cited.

2 Main Results

Let X be a sequence class. We say that a sequence class X' is *dual* of X if $X(E)' \overset{1}{=} X'(E')$ for any Banach space E, by the application $\Psi : X'(E') \longrightarrow X(E)'$ given by $\Psi\left((\varphi_j)_{j=1}^\infty\right)\left((x_j)_{j=1}^\infty\right) = \sum_{j=1}^\infty \varphi_j(x_j)$, for all $(\varphi_j)_{j=1}^\infty \in X'(E')$ and all $(x_j)_{j=1}^\infty \in X(E)$.

A *scalar sequence space* λ is a Banach space formed by scalar-valued sequences endowed with the usual coordinatewise algebraic operations, satisfying the following conditions: (i) $c_{00} \subseteq \lambda \overset{1}{\hookrightarrow} \ell_\infty$; (ii) $(e_j)_{j=1}^\infty$ is a Schauder basis for λ; (iii) $\|e_j\|_\lambda = 1$ for every $j \in \mathbb{N}$. We also define $\lambda_* := \{(\varphi(e_j))_{j=1}^\infty : \varphi \in \lambda'\}$.

Let us introduce two classes of injective type norms in the tensor product $E \otimes F$, involving sequence classes. We use the notation $\sum_{j=1}^k x_j \otimes y_j$ for any representation of $u \in E \otimes F$.

Proposition 2.1. *Let E and F be Banach spaces and λ be a scalar sequence space. Then, the expression*

$$\alpha_\lambda(u) = \sup\left\{\left|\sum_{j=1}^{k}\sum_{n=1}^{\infty}\varphi_n(x_j)\psi_n(y_j)\right|; (\varphi_n)_{n=1}^{\infty} \in B_{X(E')}, (\psi_n)_{n=1}^{\infty} \in B_{Y(F')}\right\}$$

defines a reasonable norm in $E \otimes F$ for all linearly stable classes X and Y satisfying $X(\mathbb{K}) \overset{1}{\hookrightarrow} \lambda$ and $Y(\mathbb{K}) \overset{1}{\hookrightarrow} \lambda_*$.

Proposition 2.2. *Let E and F be Banach spaces and λ be a scalar sequence space. Then, the expression*

$$\alpha'_\lambda(u) = \sup\left\{\left|\sum_{j=1}^{k}\sum_{n=1}^{\infty}\varphi_n(x_j)\psi_n(y_j)\right|; (\varphi_n)_{n=1}^{\infty} \in B_{X'(E')}, (\psi_n)_{n=1}^{\infty} \in B_{Y'(F')}\right\},$$

defines a reasonable norm in $E \otimes F$ for all linearly stable sequence classes X and Y satisfying

a) *$X'(\mathbb{K}) \overset{1}{\hookrightarrow} \lambda$ and $Y'(\mathbb{K}) \overset{1}{\hookrightarrow} \lambda_*$ or* **b)** *$X(\mathbb{K}) \overset{1}{=} \lambda$ and $Y(\mathbb{K}) \overset{1}{=} \lambda_*$, if λ is reflexive.*

Proposition 2.3. *The norms α'_λ and α_λ respects subspaces and are uniform for all scalar sequence space λ and all sequence classes X and Y satisfying the conditions of the Propositions 2.2 and 2.1. In particular, they are tensor norms.*

Once the norm α'_λ has been defined, we will characterize the dual $(E\hat{\otimes}_{\alpha'_\lambda}F)'$. Knowing that the norm α'_λ on $E \times F$ is smaller than the projective norm, every bounded linear functional on $E\hat{\otimes}_{\alpha'_\lambda}F$ is the linearization of a unique bounded bilinear form on $E \times F$. The following result gives the characterization of these bilinear forms.

Theorem 2.1. *Let $B : E \times F \longrightarrow \mathbb{K}$ be a bilinear form and X and Y sequence classes satisfying the conditions of the Proposition 2.2. Then, the linearization $\widetilde{B} : E\widehat{\otimes}_{\alpha'_\lambda}F \longrightarrow \mathbb{K}$ is continuous if and only if there exists a regular Borel measure μ on the compact $B_{X(E)'} \times B_{Y(F)'}$ such that*

$$B(x, y) = \int_{B_{X(E)'} \times B_{Y(F)'}} \sum_{n=1}^{\infty} \varphi_n(x)\psi_n(y)d\mu(\varphi, \psi) \tag{1}$$

for all $x \in E$ and $y \in F$. Besides that, $\left\|\widetilde{B}\right\| = \inf\{\|\mu\| : \mu$ satisfying (1)$\}$ and this infimum is attained.

We say that a bilinear form B on $E \times F$ is an λ-*integral bilinear form* if its linearization, $\widetilde{B}$, is a continuous linear functional on the tensor product $E\hat{\otimes}_{\alpha'_\lambda}F$. The Banach space of λ-integral bilinear forms, with the norm $\left\|\widetilde{B}\right\| = \inf\{\|\mu\|\}$, will be denoted by $\mathcal{B}_{\lambda,I}(E \times F)$. Thus we have

$$(E\hat{\otimes}_{\alpha'_\lambda}F)' = \mathcal{B}_{\lambda,I}(E \times F).$$

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DISTRIBUTIONAL CHAOS FOR CONVOLUTION OPERATORS ON $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$

J. V. PINTO & V. V. FÁVARO

[1]Universidade Federal do Rio de Janeiro, UFRJ, RJ, Brazil, joao.pinto@ufu.br
[2]Universidade Federal de Uberlândia, UFU, MG, Brazil, vvfavaro@ufu.br

Abstract

In [2], the authors defined Li-Yorke chaos in the context of topological vector spaces and they proved that all nontrivial convolution operators on the space $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ of all entire functions of infinitely many complex variables are Li-Yorke chaotic.
In this work, we will introduce the definition of distributional chaos in the context of topological vector space and we will prove that all nontrivial convolution operators on the space $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ are distributionally chaotic, improving the mentioned result obtained in [2].

1 Introduction

Let $\mathbb{C}^{\mathbb{N}}$ be the topological vector space of all complex sequences, whose topology is generated by the seminorms $(p_n)_{n\in\mathbb{N}}$, where $p_n((x_k)_k) := \max\{|x_1|, \ldots, |x_n|\}$. We say that a function $f : \mathbb{C}^{\mathbb{N}} \to \mathbb{C}$ is *entire*, if it is continuous and for each $\xi \in U$, $\eta \in E$ and $\phi \in F'$, the function

$$\lambda \mapsto \phi \circ f(\xi + \lambda\eta)$$

of one complex variable is holomorphic in a neighborhood of 0. We denote the space of all entire functions of infinitely many complex variables by $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$.
We denote by τ_0 the *compact-open* topology on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$, that is the topology generated by the seminorms

$$p_K(f) = \sup_{z\in K} |f(z)|,$$

with $K \subset \mathbb{C}^{\mathbb{N}}$ compact.

Definition 1.1. a) Let $a \in \mathbb{C}^{\mathbb{N}}$. The *translation operator by* a is the operator $\tau_a : \mathcal{H}(\mathbb{C}^{\mathbb{N}}) \to \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ defined by

$$\tau_a(f)(x) = f(x - a).$$

b) We say that a continuous linear operator $L : \mathcal{H}(\mathbb{C}^{\mathbb{N}}) \to \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is a *convolution operator* if

$$L(\tau_a f) = \tau_a(Lf),$$

for all $f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}})$ and $a \in \mathbb{C}^{\mathbb{N}}$. We say that L is *nontrivial* if it is not a scalar multiple of the identity.

In an analogous way we define convolution operators on the space $\mathcal{H}(\mathbb{C}^n)$, with $n \in \mathbb{N}$.

The definition of distributional chaos is well known in the context of metric spaces, but since we are interested in to explore the linear dynamics of convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$, which is not metrizable (see [3]), we need to explore the notion of distributional chaos for operators on topological vector spaces.

Definition 1.2. Let E be a topological vector space, B_0 be the set of all neighborhoods of 0 and $T : E \to E$ be a linear operator. We say that the subset $\Gamma \subset E$ is *distributionally scrambled* if there is $V \in B_0$ such that, for all $U \in B_0$ and for each pair of distinct points $x, y \in \Gamma$, we have
i) $\limsup_n \frac{1}{n}\text{card}\left\{0 \le i \le n-1 : T^i(x) - T^i(y)) \in U\right\} = 1$ and
ii) $\liminf_n \frac{1}{n}\text{card}\left\{0 \le i \le n-1 : T^i(x) - T^i(y)) \in V\right\} = 0$.
We say that T is *distributionally chaotic*, if E contains an uncountable and distributionally scrambled subset Γ.

2 Main Result

Consider $n \in \mathbb{N}$. We define the *canonical projection* and the *canonical inclusion*, respectively, by

$$\pi_n : (x_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mapsto (x_i)_{i=1}^{n} \in \mathbb{C}^n$$

and

$$J_n : (x_1, \cdots, x_n) \in \mathbb{C}^n \mapsto (x_1, \cdots, x_n, 0, 0 \cdots) \in \mathbb{C}^{\mathbb{N}}.$$

This two functions induce the following mappings:

$$J_n^* : f \in \mathcal{H}(\mathbb{C}^{\mathbb{N}}) \mapsto f \circ J_n \in \mathcal{H}(\mathbb{C}^n)$$

and

$$\pi_n^* : f_n \in \mathcal{H}(\mathbb{C}^n) \mapsto f_n \circ \pi_n \in \mathcal{H}(\mathbb{C}^{\mathbb{N}}).$$

We observe that the function $\pi_n^* : \mathcal{H}(\mathbb{C}^n) \to \pi_n^*(\mathcal{H}(\mathbb{C}^n))$ is a topological isomorphism. An important result about the structure of the space $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ is that $\mathcal{H}(\mathbb{C}^{\mathbb{N}}) = \bigcup_{n\in\mathbb{N}} \pi_n^*(\mathcal{H}(\mathbb{C}^n))$.
The next proposition is fundamental to prove our result. It gives the relation between convolution operators on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$ and convolution operators on $\mathcal{H}(\mathbb{C}^n)$, with $n \in \mathbb{N}$.

Proposition 2.1. *(see [2]) Let L be a convolution operator on $\mathcal{H}\left(\mathbb{C}^{\mathbb{N}}\right)$. Then:*
a) The continuous operator

$$L_n := J_n^* \circ L \circ \pi_n^* : \mathcal{H}(\mathbb{C}^n) \to \mathcal{H}(\mathbb{C}^n)$$

is a convolution operator on $\mathcal{H}(\mathbb{C}^n)$.
b) $L(f_n \circ \pi_n) = (L_n f_n) \circ \pi_n$ for all $f_n \in \mathcal{H}(\mathbb{C}^n)$ and $n \in \mathbb{N}$.
c) L is a multiple scalar of identity on $\mathcal{H}\left(\mathbb{C}^{\mathbb{N}}\right)$ if, and only if, L_n is a multiple scalar of identity on $\mathcal{H}(\mathbb{C}^n)$, for all $n \in \mathbb{N}$.

Our main result in this work is the following:

Theorem 2.1. *If L is a nontrivial convolution operator on $\mathcal{H}(\mathbb{C}^{\mathbb{N}})$, then L is distributionally chaotic.*

Sketch of the proof: Since L is nontrivial, it follows from Proposition 2.1 (c) that there is $n \in \mathbb{N}$ such that L_n is a nontrivial convolution operator on $\mathcal{H}(\mathbb{C}^n)$.

Since nontrivial convolution operators on $\mathcal{H}(\mathbb{C}^n)$ are distributionally chaotic (see [1]), then there exists an uncountable and distributionally scrambled subset $D \subset \mathcal{H}(\mathbb{C}^n)$.

Using the last proposition and the fact that the function $\pi_n^* : \mathcal{H}(\mathbb{C}^n) \to \pi_n^*(\mathcal{H}(\mathbb{C}^n))$ is a topological isomorphism, it is possible to prove that the subset $\pi_n^*(D)$ is distributionally scrambled for L. $\square$

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ON B-CLASSES AND COINCIDENCE RESULTS IN OPERATOR THEORY

J. R. CAMPOS & L. F. P. SOUSA

[1]Universidade federal da Paraíba, DCX, PB, Brazil, jamilson@dcx.ufpb.br

[2]Universidade federal da Paraíba, DM, PB, Brazil, lfpinhosousa@gmail.com

Abstract

From a Banach space prefixed B and a sequence class X, we define a sequence class called B-class associated with X that generalizes some well-known vector-valued sequence spaces. Examples and results on B-classes are presented and a class of coincidence results in the theory of operator ideals is given.

1 Introduction

In [3], Fourie and Zeekoei proved that the ideal formed by operators mapping weakly p-summable sequences to operator p-summable sequences is complete with a certain norm. In this direction, from a fixed Banach space Y, they present the space of all operator $[Y, p]$-summable sequences, with generalize operator p-summable sequences (operator $[\ell_p, p]$-summable with this notation), defined by Karn and Sinha in [4], and proving some facts about this space.

From the perspective of the abstract environment of sequence classes that was developed by Botelho and Campos in [1], we present a sequence class $X^B(\cdot)$ from a sequence class X and a prefixed Banach space B what we call B-class associated with X. Depending on the choices for X and B, for each Banach space E, the space $X^B(E)$ generalizes some sequence spaces as the spaces of $[Y, p]$-summable and operator p-summable sequences. We will explore some properties of this sequence class and show coincidence results in operator theory that arises from this construction.

The space of operator p-summable sequences was renamed by Botelho, Campos and Santos (in [2]) to mid p-summable sequences and denoted by $\ell_p^{\text{mid}}(E)$, the space of all mid p-summable sequences with values in E. The letters B, E and F will denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathcal{L}(E; F)$ the Banach space of bounded linear operators from E to F with the usual operator norm.. S_E and E' will represent the unit sphere and the topological dual of E, respectively. When we write $E \overset{1}{\hookrightarrow} F$ will mean that E is a subspace of F and $\|\cdot\|_F \leq \|\cdot\|_E$. Moreover, $E \overset{1}{=} F$ means that E and F are isometrically isomorphic. The expressions $E \mapsto X(E)$, $X(\cdot)$ or simply X, as long as there is no risk of doubt, represent a sequence class. The other symbols and terminology regarding sequence classes can be found in [1].

2 Main Results

Let X be a sequence class X and B be a Banach space. We define the normed space

$$X^B(E) := \{(x_j)_{j=1}^\infty \in E^{\mathbb{N}} : (T(x_j))_{j=1}^\infty \in X(B), \forall T \in \mathcal{L}(E; B)\},$$

with norm

$$\|(x_j)_{j=1}^\infty\|_{X^B(E)} := \sup_{T \in B_{\mathcal{L}(E;B)}} \|(T(x_j))_{j=1}^\infty\|_{X(B)}.$$

When X is finitely determined (which will henceforth be taken as a hypothesis), we show that $X^B(E)$ is a Banach space and we can verify that $E \mapsto X^B(E)$ define a sequence class called *B-class associated with X*. When X is linearly stable we have $X \overset{1}{\hookrightarrow} X^B$.

Immediate examples of sequence classes of this nature are $\ell_p^{\mathbb{K}}(\cdot) = \ell_p^w(\cdot)$ and $\ell_p^{\ell_p}(\cdot) = \ell_p^{\mathrm{mid}}(\cdot)$. We have $\ell_\infty^B(\cdot) = \ell_\infty(\cdot)$ for all fixed Banach space B, and we define

$$X^w(\mathbb{E}) := X^{\mathbb{K}}(E) = \{(x_j)_{j=1}^\infty \in E^{\mathbb{N}} : (\varphi(x_j))_{j=1}^\infty \in X(\mathbb{K}), \forall \varphi \in E'\}.$$

Definition 2.1. We say that a sequence class X is *spherically injective*, if for each $x \in S_E$ and $(\alpha_j)_{j=1}^\infty \in \mathbb{K}^{\mathbb{N}}$, $(\alpha_j)_{j=1}^\infty \in X(\mathbb{K})$ whenever $(x\alpha_j)_{j=1}^\infty \in X(E)$, and $\|(\alpha_j)_{j=1}^\infty\|_{X(\mathbb{K})} \leq \|(x\alpha_j)_{j=1}^\infty\|_{X(E)}$.

Some examples of spherically injective sequence classes are ℓ_p, ℓ_p^w, ℓ_∞, ℓ_p^{mid}, $\ell_p\langle\cdot\rangle$ and ℓ_M, where M is an Orlicz function satisfying $M(1) = 1$.

If X is a spherically injective class, then $X^B(\mathbb{K}) \overset{1}{\hookrightarrow} X(\mathbb{K})$ which together with the hypothesis of linear stability of X gives us $X^B(\mathbb{K}) \overset{1}{=} X(\mathbb{K})$. Then a result from the sequence classes theory [1, Theorem 3.6] ensures that $\mathcal{L}_{X^B;X}(E;F)$ is a Banach operator ideal.

Theorem 2.1. *Let B and F be Banach spaces and X be a linearly stable and spherically injective sequence class. Then, for all Banach space E, $\mathcal{L}_{X^F;X}(E;F) \overset{1}{=} \mathcal{L}_{X^B;X}(E;F) \overset{1}{=} \mathcal{L}(E;F)$ if and only if, $X^B(E) \overset{1}{\hookrightarrow} X^F(E)$.*

Corollary 2.1. *Let B be a Banach space. If X is a linearly stable and spherically injective sequence class, then $X^B \overset{1}{\hookrightarrow} X^w$, that is, $X^B(E) \overset{1}{\hookrightarrow} X^w(E)$ for all Banach space E.*

Some examples of known and new inclusion results obtainad by the Theorem 2.1:
a) Taking $X(\cdot) = \ell_p(\cdot)$, $B = \mathbb{K}$ and $F = \ell_p$, we obtain

$$\Pi_p^{\mathrm{mid}}(E;\ell_p) \overset{1}{=} \Pi_p(E;\ell_p) \overset{1}{=} \mathcal{L}(E;\ell_p) \Leftrightarrow \ell_p^w(E) \overset{1}{\hookrightarrow} \ell_p^{\mathrm{mid}}(E).$$

This coincidence can be found in [2].
b) This is a new result: talking $X(\cdot) = \ell_M(\cdot)$, $B = \mathbb{K}$ and $F = \ell_M$, where M is a Orlicz function such that $M(1) = 1$, we obtain

$$\mathcal{L}_{\ell_M^{\mathrm{mid}};\ell_M}(E;\ell_M) \overset{1}{=} \mathcal{L}_{\ell_M^w;\ell_M}(E;\ell_M) \overset{1}{=} \mathcal{L}(E;\ell_M) \Leftrightarrow \ell_M^w(E) \overset{1}{\hookrightarrow} \ell_M^{\mathrm{mid}}(E).$$

Here, we consider $\ell_M^{\mathrm{mid}}(E)$ the space defined in [5], which defines the sequence class $\ell_M(\cdot)$.

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ZERO SETS OF HOMOGENEOUS POLYNOMIALS CONTAINING INFINITE DIMENSIONAL SPACES

MIKAELA AIRES[1] & GERALDO BOTELHO[2]

[1]Universidade de São Paulo, IME, SP, Brazil, mikaela-aires@ime.usp.br

[2]Universidade Federal de Uberlândia, IME, MG, Brazil, botelho@ufu.br

Abstract

Let X be a (real or complex) infinite dimensional linear space. We establish conditions on a homogeneous polynomial P on X so that, if W is any finite dimensional subspace of X on which P vanishes, then P vanishes on an infinite dimensional subspace of X containing W. In the complex case, this is a step beyond the classical result due to Plichko and Zagorodnyuk. Applications to the real case are also provided.

1 Introduction

In 1998, Plichko and Zagorodnyuk [7] proved the following remarkable result: For any infinite dimensional complex linear space X, every $\mathbb{C}$-valued homogeneous polynomial on X vanishes on an infinite dimensional subspace of X. The real case of the problem, which is clearly very different from the complex case, was thoroughly studied by several authors, see, e.g., [3, 4].

In the modern language of lineability (see [2]), the Plichko-Zagorodnyuk theorem asserts that the zero set of any homogeneous polynomial on an infinite dimensional complex space is lineable, meaning that it contains an infinite dimensional linear space. In this work, we investigate the problem solved by Plichko-Zagorodnyuk in the complex case under the perspective of the notions introduced in [5] and developed in, e.g., [1]. More precisely, we are interested in the following question:

Given a homogeneous polynomial P on a (real or complex) infinite dimensional linear space X and given a finite dimensional subspace W of X on which P vanishes, does P vanish on an infinite dimensional subspace of X containing W?

The following definition is given just for the sake of simplicity.

Definition 1.1. A nonempty subset A of an infinite dimensional linear space X is *finitely lineable* if, for every finite dimensional subspace of X contained in $A \cup \{0\}$, there exists an infinite dimensional subspace of X containing W and contained in $A \cup \{0\}$.

Note that every finitely lineable set contains, up to 0, an infinite dimensional space, that is, it is lineable. Since $P(0) = 0$ for every homogeneous polynomial P, the answer to the question above is affirmative if and only if the zero set of P is finitely lineable.

Let X be an infinite dimensional linear space $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Given an m-homogeneous polynomial $P\colon X \longrightarrow \mathbb{K}$, by $\check{P}\colon X^m \longrightarrow \mathbb{K}$ we denote the (unique) symmetric m-linear form associated to P, that is, $P(x) = \check{P}(\underbrace{x, \ldots, x}_{m \text{ times}})$ for every $x \in X$. Given $0 \leq k \leq m$, $x_1, \ldots, x_k \in X$, and $\alpha_1, \ldots, \alpha_k \in \{0, 1, \ldots, k\}$ with $\alpha_1 + \cdots + \alpha_k = m$, we shall use the simplified notation $\check{P}(x_1^{\alpha_1}, \ldots, x_k^{\alpha^k}) := \check{P}(\underbrace{x_1, \ldots, x_1}_{\alpha_1 \text{ times}}, \ldots, \underbrace{x_k, \ldots, x_k}_{\alpha_k \text{ times}})$.

Definition 1.2. Let $P\colon X \longrightarrow \mathbb{K}$ be an m-homogeneous polynomial. For a t-homogeneous polynomial $Q\colon X \longrightarrow \mathbb{K}$, $1 \leq t \leq m-1$, we write $Q \prec P$ if there are $x_1, \ldots, x_n \in X$ on which P vanishes, and $\alpha_1, \ldots, \alpha_n \in \mathbb{N} \cup \{0\}$ with $\alpha_1 + \cdots + \alpha_n + t = m$ such that $Q(x) = \check{P}(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}, x^t)$ for every $x \in X$.

2 Main Results

The main result of this work establishes conditions on a homogeneous polynomial under which the answer to the question above is affirmative.

Theorem 2.1. *Let $P\colon X \longrightarrow \mathbb{K}$ be an m-homogeneous polynomial. Suppose that, for every infinite dimensional subspace Y of X, P vanishes on some nonzero vector of Y and every homogeneous polynomial $Q \prec P$ vanishes on an infinite dimensional subspace of Y. Then the zero set of P is finitely lineable.*

As a first application, we have the following result.

Corollary 2.1. *Let $P_1, \ldots, P_k$ be homogeneous polynomials on X. If, for every infinite dimensional subspace Y of X, each P_i vanishes on a nonzero vector of Y and each homogeneous polynomial $Q \prec P_i, i = 1, \ldots, k$, vanishes on an infinite dimensional subspace of Y, then the set $\bigcap\limits_{i=1}^{k} P_i^{-1}(0)$ is finitely lineable.*

A particular case of Theorem 2.1 gives a contribution to the subject of pointwise lineability, introduced in [6]:

Corollary 2.2. *Suppose that a homogeneous polynomial P on X satisfying the assumptions of Theorem 2.1 vanishes on a point x of X. Then there is an infinite dimensional subspace of X containing x and contained in the zero set of P.*

In the complex case, the following corollary is an extension of the Plichko-Zagorodnyuk theorem [7].

Corollary 2.3. *The zero set of any homogeneous polynomial on an infinite dimensional complex linear space is finitely lineable.*

Again, a particular case gives a contribution to pointwise lineability:

Corollary 2.4. *Suppose that a homogeneous polynomial P on a complex infinite dimensional linear space X vanishes on a point $x \in X$. Then there is an infinite dimensional subspace of X containing x and contained in the zero set of P.*

The next result illustrates how our results can be applied in the real case.

Proposition 2.1. *The zero set of any homogeneous polynomial of finite type on any infinite dimensional (real or complex) linear space is finitely lineable.*

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SPACEABILITY OF QUASI-BANACH OPERATORS MULTI-IDEALS

N. G. ALBUQUERQUE[1] & L. COLETA[1]

[1]Universidade Federal da Paraíba, DM, PB, Brazil,
nacib.albuquerque@academico.ufpb.br, lindinescoleta@gmail.com

Abstract

Spaceability results related to (quasi)-Banach multilinear operators are provided. Also we investigate applications on new and classical summing operators classes.

1 Introduction

Throughout this $E_1, \ldots, E_m, E, F$ will be Banach or quasi-Banach spaces and, as usual, the topological dual and the closed unit ball of E will be denote by E' and B_E, respectively, for $E' \neq \{0\}$. We will denote by $\mathcal{L}(E_1, \ldots, E_m; F)$ the Banach (or quasi-Banach) space of bounded m-linear operators from $E_1 \times \cdots \times E_m$ to F endowed with the usual sup norm (quasi-norm). Also, we wiil denote by $\mathcal{M}$ a *quasi-Banach multi-ideal.*

We continue the search of large topological structures in the general framework of multilinears operators ideals, in the sense of Pietsch [3]. Hernandéz et al. [4] investigated whenever $\mathcal{I}_1(E;F) \setminus \mathcal{I}_2(E;F)$ is spaceable, for ideals $\mathcal{I}_1, \mathcal{I}_2$ of bounded linear operators on Banach space with certain properties, providing a result that encompasses several others related with spaceability problems of many linear operator ideals. We take a step further and investigate this problem in both multilinear and quasi-Banach spaces, obtaining a general result. First we recall a notion originally introduced in [4] for Banach spaces.

Definition 1.1 ($\sigma-$reproducible space)**.** *A (quasi)-Banach space is said to be $\sigma-$*reproducible *if there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of complemented subspaces; and for each $n \in \mathbb{N}$, there are $P_n : E \to E_n$ a bounded projection, with $P_i \circ P_j = 0$ if $i \neq j$, and $\phi_n : E_n \longrightarrow E$ a isomorphism. Moreover, for all $k \in \mathbb{N}$ the projections $\widetilde{P}_k = \sum_{n=1}^{k} P_n : E \longrightarrow \bigoplus E_n$ are uniformily bounded.*

In [4] is presented a number of nice properties fulfilled by $\sigma-$reproducible Banach spaces, as well illustrative examples. For instance, the class of all Banach spaces E which are isomorphic to the vector valued sequence space $c_0(E)$ or $\ell_p(E)$ for $1 \leq p < \infty$. It is plain that this is also true when we consider quasi-Banach spaces E and $0 < p < 1$.

We provide spaceability results on the context of quasi-Banach multi-ideals. The techniques follow along the lines of Hernandéz et al. [4].

2 Main Results

The main results are presented next.

Theorem 2.1. *Let $E_1, \ldots, E_m, F$ be quasi-Banach spaces and let $\mathcal{M}_1, \mathcal{M}_2$ be operators quasi-normed multi-ideals. If E_j, for some $j = 1, \ldots, m$, or F is a $\sigma-$reproducible space, $\mathcal{M}_1(E_1 \ldots, E_m; F)$ is a quasi-Banach multi-ideal, and $\mathcal{M}_1(E_1 \ldots, E_m; F) \setminus \mathcal{M}_2(E_1 \ldots, E_m; F)$ is non-empty, then $\mathcal{M}(E_1 \ldots, E_m; F) \setminus \bigcup_{n=1}^{\infty} \mathcal{M}_n(E_1 \ldots, E_m; F)$ is spaceable.*

Theorem 2.2. *Let $E_1, \ldots, E_m, F$ be quasi-Banach spaces and let $\mathcal{M}, \mathcal{M}_n, n \in \mathbb{N}$ be operators quasi-normed multi-ideals. If E_j, for some $j = 1, \ldots, m$, or F is a $\sigma-$reproducible space, $\mathcal{M}(E_1 \ldots, E_m; F)$ is a quasi-Banach multi-ideal, and for all $n \in \mathbb{N}$ $\mathcal{M}(E_1 \ldots, E_m; F) \setminus \mathcal{M}_n(E_1 \ldots, E_m; F)$ is non-empty, then $\mathcal{M}(E_1 \ldots, E_m; F) \setminus \bigcup_{n=1}^{\infty} \mathcal{M}_n(E_1 \ldots, E_m; F)$ is spaceable.*

As applications we provide the following (see [1]).

Corollary 2.1. *Let $\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u} \in [1, +\infty)^m$, let $\Lambda \subset \Gamma \subset \mathbb{N}^m$ be sets of indexes. Then*

$$\Pi^{\Lambda}_{(\mathbf{r},\mathbf{s})}\left(E_1, \ldots, E_m; \mathcal{E}\right) \setminus \Pi^{\Gamma}_{(\mathbf{t},\mathbf{u})}\left(E_1, \ldots, E_m; \mathcal{E}\right)$$

is either empty or $\mathfrak{c}$-spaceable, where Π^{Λ}, Π^{Γ}, denotes the class of Λ-summing, Γ-summing, operators, respectively.

Corollary 2.2. *Let $q \in (0, \infty]$ and $\mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u} \in [1, +\infty)^m$. Then each one of the sets*

$$\mathcal{L}\left(E_1, \ldots, E_m; \ell_q\right) \setminus \Pi^{ms}_{(\mathbf{t},\mathbf{u})}\left(E_1, \ldots, E_m; \ell_q\right), \quad \mathcal{L}\left(E_1, \ldots, E_m; \ell_q\right) \setminus \Pi^{as}_{(\mathbf{t},\mathbf{u})}\left(E_1, \ldots, E_m; \ell_q\right)$$

and

$$\Pi^{as}_{(\mathbf{r},\mathbf{s})}\left(E_1, \ldots, E_m; \ell_q\right) \setminus \Pi^{ms}_{(\mathbf{t},\mathbf{u})}\left(E_1, \ldots, E_m; \ell_q\right)$$

is either empty or $\mathfrak{c}$-spaceable, where Π^{as} and Π^{ms} stand for the class of absolutely *and* multiple *multilinear summing classes.*

Corollary 2.3. *Let $1 \leq q \leq \infty$ and $1 \leq r < s < \infty$. Then*

$$\Pi_{(r,s)}(E, \ell_q) \setminus DP(E, \ell_q)$$

is either empty or $\mathfrak{c}$-spaceable, where DP stands for the Dunford-Pettis operators class.

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HAMILTON-JACOBI-BELLMAN EQUATION FOR OPTIMAL CONTROL PROBLEMS WITH UNCERTAINTY

OSCAR SIERRA FONSECA[1], M. SOLEDAD ARONNA[1] & MICHELE PALLADINO[2]

[1]Escola de Matemática Aplicada, Fundação Getulio Vargas, RJ, Brazil,
oscar.fonseca@fgv.br, soledad.aronna@fgv.br

[2] University of L'Aquila - Department of Information Engineering, Computer Science and Mathematics, L'Aquila, Italy,
michele.palladino@univaq.it

Abstract

We study the properties of the value function associated with an optimal control problem with uncertainties, known as *average* or *Riemann-Stieltjes* problem. Uncertainties are assumed to belong to a compact metric probability space, and appear in the dynamics, in the terminal cost and in the initial condition, which yield an infinite-dimensional formulation. By stating the problem as an evolution equation in a Hilbert space, we show that the value function is the unique lower semi-continuous proximal solution of the Hamilton-Jacobi-Bellman (HJB) equation. Our approach relies on invariance properties and the dynamic programming principle.

1 Introduction

This work aims to prove that the value function for Mayer's problem, defined in a Hilbert space, is the unique lower semi-continuous solution of the Hamilton-Jacobi-Bellman equation when the nonlinear dynamics are measurable in time and the cost is an integral functional. Specifically, we investigate the parametrized Riemann-Stieltjes problem denoted by $(P)_{s,\varphi}$, concerning the initial time s and the initial states, which are ω-dependent and represented by the mapping $\varphi \in L^2(\mu, \Omega; \mathbb{R}^n)$:

$$\min \int_\Omega g(x(T,\omega),\omega)d\mu(\omega),$$

s.t.

$$\begin{cases} \dot{x}(t,\cdot) = f(t, x(t,\cdot), u(t), \cdot), & \text{a.e. } t \in [s,T], \\ x(s,\cdot) = \varphi(\cdot), \\ u(t) \in \mathbf{U}, \quad \text{a.e. } t \in [s,T], & \mathbf{U} \subset \mathbb{R}^m \quad \text{compact}, \end{cases} \tag{1}$$

for every $\omega \in \Omega$ and $s \le t \le T$, where (Ω, d_Ω, μ) is a compact metric measure space, with control $u \in L^\infty(0,T;\mathbf{U})$.

Initially, following the approach used in [2, 3, 3] for the semilinear case, we establish, under certain hypotheses on the dynamics (referred to as $\mathbf{H}$ and $\mathbf{C}$) and on the measure μ (referred to as $\mathbf{H}_\mu$), that the set of trajectories defined by

$$S_{[s,T]}(\varphi) := \left\{ x \in C([s,T] : L^2(\mu,\Omega;\mathbb{R}^n)) : \ x \text{ solves (1) and } \ x(s,\cdot) = \varphi(\cdot) \right\}$$

is compact in an appropriate space of functions. Subsequently, we provide a characterization of the lower semicontinuity of the associated value function for the problem $(P)_{s,\varphi}$, defined by

$$V(s,\varphi) = \inf\left\{ \int_\Omega g(x(T,\omega),\omega)d\mu(\omega) : x \in S_{[s,T]}(\varphi) \right\}$$

which establishes the existence of optimal trajectories.

The existence of minimizers, combined with invariance principles and the Dynamic Programming Principle, will pave the way to prove that the value function defined above is the unique lower semicontinuous solution of the following Hamilton-Jacobi-Bellman equation, defined in an infinite-dimensional space.

$$\begin{cases} -\left[V_t(t,\varphi) + H(t,\varphi,V_\varphi(t,\varphi))\right] = 0, \\ V(T,\varphi) = \displaystyle\int_\Omega g\left(\varphi(\omega),\omega\right) d\mu(\omega), \end{cases}$$

where $H : [0,T] \times L^2(\mu,\Omega;\mathbb{R}^n) \times L^2(\mu,\Omega;\mathbb{R}^n)^* \to \mathbb{R}$ is the Hamiltonian function given by

$$H(t,\varphi,p) := \inf_{u(t)\in\mathbf{U}} \left\langle p, f(t,\varphi(\cdot),u(t),\cdot)\right\rangle .$$

The proof strategy employs the differential inclusion approach and utilizes some results from [1]. Specifically, we define the set-valued map

$$F : [0,T] \times L^2(\mu,\Omega;\mathbb{R}^n) \rightsquigarrow L^2(\mu,\Omega;\mathbb{R}^n)$$

given by $F(t,\varphi) = f(t,\varphi,U(t),\cdot)$. The associated differential inclusion is then expressed as:

$$\dot{x}(t,\cdot) \in F(t,x(t,\cdot)) \quad \text{a.e. } t\in[s,T] \quad \text{with} \quad x(s,\omega)=\varphi(\omega).\ \ \forall\,\omega\in\Omega.$$

Finally, we prove the principal result of this work:

2 Main Result

Theorem 2.1. *Let us assume that* $(\mathbf{H})$, $(\mathbf{H}_\mu)$ *and* ***(C)*** *hold true. Then the value function* V *of problem* $(P)_{s,\varphi}$ *is the unique lower semi-continuous, bounded below function such that there exists a set* $I \subset [0,T)$ *of full measure for which, for every* $(t,\varphi,\alpha) \in \mathrm{epi}V \cap \left(I \times L^2(\mu,\Omega;\mathbb{R}^n)\times\mathbb{R}\right)$, *one has*

$$\xi_0 + \min_{v\in F(t,\varphi)} \langle v,\xi\rangle = 0 \qquad \forall\ (\xi_0,\xi,-q) \in N^P_{\mathrm{epi}V}(t,\varphi,\alpha),$$

$$V(T,\varphi) = \int_\Omega g\left(\varphi(\omega),\omega\right) d\mu(\omega).$$

Where $(\xi_0,\xi) \in \partial_P V(t,\varphi)$, *the proximal subdifferential, or* P*-subdifferential of* V.

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COMPLEX SYMMETRY OF LINEAR FRACTIONAL COMPOSITION OPERATORS ON A HALF-PLANE

V. V. FÁVARO[1], P. V. HAI[2] & O. R. SEVERIANO[3]

[1]Universidade Federal de Uberlândia, UFU, MG, Brazil, vvfavaro@ufu.br
[2]Hanoi University of Science and Technology, Vietnam, hai.phamviet@hust.edu.vn
[3]Universidade Estadual de Campinas, IMECC, SP, Brazil, orreiss@unicamp.br

Abstract

We investigate the bounded composition operators induced by linear fractional self-maps of the right half-plane $\mathbb{C}_+$ on the Hardy space $H^2(\mathbb{C}_+)$. We completely characterize which of these operators are cohyponormal and we find conjugations for the linear fractional composition operators that are complex symmetric.

1 Introduction

All along this work, $\mathbb{C}$ denotes the complex plane, $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk and $\mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ is the right half-plane.

Let $\Omega \subset \mathbb{C}$ and let $\mathcal{S}$ be a space of functions defined on Ω. A composition operator C_ϕ on $\mathcal{S}$ is an operator acting by composition to the right with a chosen self-map ϕ of Ω, i.e.,

$$C_\phi f = f \circ \phi, \quad f \in \mathcal{S}.$$

The self-map ϕ is called the *symbol* of the composition operator C_ϕ. If $\Omega = \mathbb{C}_+$, Elliot and Jury established a boundedness criterion for composition operators on the Hardy space of the right half-plane $H^2(\mathbb{C}_+)$ in terms of angular derivative (see [1, Theorem 3.1]). As a consequence of this caracterization, we have that the only linear fractional self-maps of $\mathbb{C}_+$ inducing bounded composition operators on $H^2(\mathbb{C}_+)$ are those of the following form

$$\phi(w) = aw + b, \quad \text{where } a > 0 \text{ and } \mathrm{Re}(b) \geq 0. \tag{1}$$

Let $\mathcal{L}(\mathcal{H})$ denote the space of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. A *conjugation* on $\mathcal{H}$ is a conjugate-linear operator satisfying $C^2 = I$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *cohyponormal* if $\|Tx\| \leq \|T^*x\|$ for all $x \in \mathcal{H}$, *normal* if $TT^* = T^*T$, *self-adjoint* if $T = T^*$, *unitary* if $TT^* = I = T^*T$ and *complex symmetric* (or C-*symmetric*) if there is a conjugation C on $\mathcal{H}$, for which $CT^*C = T$. In [3], Noor and Severiano studied the composition operators on $H^2(\mathbb{C}_+)$ induced by linear fractional self-maps of $\mathbb{C}_+$. They completely characterize the symbols that induce complex symmetric composition operators (see next theorem) and provide a new prove to characterize normal, self-adjoint and unitary composition operators on $H^2(\mathbb{C}_+)$.

Theorem 1.1. *[3, Theorems 2 and 6] Let ϕ be as in (1). Then*

1. *C_ϕ is normal on $H^2(\mathbb{C}_+)$ if and only if $a = 1$ or $\mathrm{Re}(b) = 0$.*

2. *C_ϕ is self-adjoint on $H^2(\mathbb{C}_+)$ if and only if $a = 1$ and $b \geq 0$.*

3. *C_ϕ is unitary on $H^2(\mathbb{C}_+)$ if and only if $a = 1$ and $\mathrm{Re}(b) = 0$.*

4. *C_ϕ is complex symmetric on $H^2(\mathbb{C}_+)$ if and only if C_ϕ is normal on $H^2(\mathbb{C}_+)$.*

Noor and Severiano [3] study the complex symmetry of composition operators on $H^2(\mathbb{C}_+)$. The key of this study is to analyze the cyclic behavior of the bounded composition operators induced by linear fractional self-maps of $\mathbb{C}_+$, which allowed them to characterize the operators that are complex symmetric. Despite this characterization, they did not exhibit the conjugations for the cases that these operators are complex symmetric.

2 Main Results

In this section, we present the main results we obtained in [2].

Theorem 2.1. *Let ϕ be the self-map of $\mathbb{C}_+$ defined by $\phi(w) = aw + b$.*

(a) If $a \in (0, 1)$ and $\mathrm{Re}(b) > 0$, then C_ϕ is not cohyponormal.

(b) If $a \in (1, \infty)$ and $\mathrm{Re}(b) > 0$, then C_ϕ is cohyponormal.

From Theorem 2.1, we obtain the complete characterization of the linear fractional composition operators that are complex symmetric.

Corollary 2.1. *For $a > 0$ and $\mathrm{Re}(b) \geq 0$, let ϕ be the self-map of $\mathbb{C}_+$ defined by $\phi(w) = aw + b$. Then C_ϕ is complex symmetric on $H^2(\mathbb{C}_+)$ if and only if $a = 1$ or $\mathrm{Re}(b) = 0$.*

Next we deal with the problem of exhibit conjugations for the linear fractional composition operators that are complex symmetric.

Theorem 2.2. *Let ϕ be an analytic self-map of $\mathbb{C}_+$ such that C_ϕ is bounded on $H^2(\mathbb{C}_+)$. Then C_ϕ is complex symmetric with respect to the conjugation $Jf(w) = \overline{f(\overline{w})}$ if, and only if, ϕ has the form $\phi(w) = w + b$.*

Theorem 2.3. *Let $\phi(w) = aw + b$ with $a \in (0, 1) \cup (1, \infty)$ and $\mathrm{Re}(b) = 0$. If $\tau(w) = w + (a - 1)^{-1}b$, $(Wf)(w) = \frac{1}{w}\overline{f\left(\frac{1}{\overline{w}}\right)}$ and $W_{a,b} := C_\tau^{-1} W C_\tau$, then*

(a) W and $W_{a,b}$ are conjugations.

(b) C_ϕ is $W_{a,b}$-symmetric on $H^2(\mathbb{C}_+)$.

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ON COMPACT SETS OF C_0-SUM SPACES

T. GRANDO

Midwestern Paraná State University, UNICENTRO, PR, Brazil, tgrando@unicentro.com

Abstract

The main purpose of this talk is to study holomorphic mappings of bounded type from $c_0\left(\bigoplus_{i=1}^{\infty}\ell_p^i\right)$, $1 \leq p < \infty$, into a complex Banach space Y. To do this, we define a fundamental system of compact sets of $c_0\left(\bigoplus_{i=1}^{\infty}\ell_p^i\right)$, $1 \leq p < \infty$. This is a joint work with Mary Lilian Lourenço.

1 Introduction

Let X be a Banach space. A subset $A \subset X$ is called totally bounded if for every $\epsilon > 0$ there are $x_1, \ldots, x_n \in X$ such that $A \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$. In this context, a set $K \subset X$ is compact if and only if K is complete and tottally bounded. R. Ryan, in [3], described the holomorphic mappings of bounded type from the Banach space ℓ_1 into a complex Banach space Y. For this, a simple characterization of the compact subsets of ℓ_1 was used: $K \subset \ell_1$ is relatively compact if and only if $\lim_{n\to\infty}\sum_{k=n}^{\infty}|z_k| = 0$ uniformly in $z \in K$. With this characterization, a fundamental system of compact sets of ℓ_1 was defined, allowing the achievement of the main result.

In this work, we are interested in studying holomorphic mappings of bounded type from the Banach space $c_0\left(\bigoplus_{i=1}^{\infty}\ell_p^i\right)$ into a complex Banach space Y. To do this, we will present a fundamental system for the compact sets of $c_0\left(\bigoplus_{i=1}^{\infty}\ell_p^i\right)$. We recall that a fundamental system of compact sets of X is a family of compact sets $\{A_\lambda : \lambda \in I\} \subset X$ such that , for each compact $K \subset X$, there is $\lambda \in I$ such that $K \subset A_\lambda$.

2 Main Results

Let c_0 denote the space of null sequences, $c_0^+ = \{(\lambda_j) \in c_0 : \lambda_j > 0\}$ and for all $1 \leq p < \infty$, $c_0\left(\bigoplus_{i=1}^{\infty}\ell_p^i\right) = \left\{(y_i)_i \in \mathbb{K}^{\mathbb{N}} : \left(\left(\sum_{i\in I(n)}|y_i|^p\right)^{1/p}\right)_n \in c_0\right\}$, with the norm $y = \sup_{n\in\mathbb{N}}\left(\sum_{i\in I(n)}|y_i|^p\right)^{1/p}$.

Proposition 2.1. *For each $\lambda = (\lambda_m) \in c_0^+$ and $1 \leq p < \infty$, consider*

$$A_\lambda = \left\{(y_i) \in c_0\left(\bigoplus_{i=1}^{\infty}\ell_p^i\right) : \quad \left(\sum_{i\in I(m)}|y_i|^p\right)^{1/p} \leq \lambda_m \quad m \in \mathbb{N}\right\}.$$

Then the polydiscs $\left\{A_\lambda : \lambda \in c_0^+\right\}$ form a fundamental system of compact sets of $c_0\left(\bigoplus_{i=1}^{\infty}\ell_p^i\right)$.

Proof For each $1 < p < \infty$, let $\lambda \in c_0^+$ and $\epsilon > 0$. We claim that A_λ is closed in $c_0\left(\bigoplus_{i=1}^{\infty}\ell_p^i\right)$. For each $y \in \overline{A_\lambda}$ there is a sequence $(y^j) \subset A_\lambda$ such that $y^j \to y$. Then there exist $j_0 \in \mathbb{N}$ such that $\|y^j - y\| \leq \epsilon$, for all $j \geq j_0$. Therefore $\left(\sum_{i\in I(m)}|y_i|^p\right)^{1/p} \leq \lambda_m, \quad$ for all $\quad m \in \mathbb{N}$. Thus, A_λ is closed.

Now, we claim that A_λ is compact. It is sufficient to prove that A_λ is totally bounded in $c_0\left(\bigoplus_{i=1}^{\infty}\ell_p^i\right)$. Let $\epsilon > 0$. By definition, there is $N \in \mathbb{N}$ such that $\lambda_m \leq \frac{\epsilon}{2}$ for all $m \geq N$. We define the compact set

$A_\lambda^N = \{(y_i) \in A_\lambda : y_i = 0 \quad \text{for all} \quad i \in I(m) \quad \text{e} \quad m > N\}$. Therefore $A_\lambda^N = T(F)$ where $T : \mathbb{C}^n \longrightarrow c_0\left(\bigoplus_{i=1}^\infty \ell_p^i\right)$ is a continuous mapping defined by

$$T(z_1, ..., z_n) = \begin{pmatrix} z_1 & z_2 & z_4 & \dots & z_k & 0 & \dots \\ & z_3 & z_5 & & & 0 & \\ & & z_6 & & & 0 & \\ & & & \ddots & \vdots & \vdots & \\ & & & & z_n & 0 & \\ & & & & & 0 & \end{pmatrix},$$

and $F = \left\{ z \in \mathbb{C}^n : \left(\sum_{i \in I(m)} |z_i|^p\right)^{1/p} \leq \lambda_m, \quad 1 \leq m \leq N \right\}$ is a compact set of $\mathbb{C}^n$, for specific $n = \frac{N(N+1)}{2}$. Therefore, there are $z^1, ..., z^l \in A_\lambda^N$, such that $A_\lambda^N \subset \bigcup_{j=1}^l B(z^j, \frac{\epsilon}{2})$.

If $y \in A_\lambda$ we can write $y = v + w$, where

$$v = \begin{pmatrix} y_1 & y_2 & y_4 & \dots & y_k & 0 & \dots \\ & y_3 & y_5 & & & 0 & \\ & & y_6 & & & 0 & \\ & & & \ddots & \vdots & \vdots & \\ & & & & y_m & 0 & \\ & & & & & 0 & \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & y_{m+1} & \dots \\ & 0 & 0 & & & y_{m+2} & \\ & & 0 & & & y_{m+3} & \\ & & & \ddots & \vdots & \vdots & \\ & & & & 0 & & \\ & & & & & & \end{pmatrix}.$$

Since $v \in A_\lambda^N$ there is $1 \leq j_0 \leq l$ such that $v \in B(z^{j_0}, \frac{\epsilon}{2})$. Besides that, $\|w\| \leq \frac{\epsilon}{2}$ and $\|y - z^{j_0}\| = \|v + w - z^{j_0}\| \leq \epsilon$ and as a consequence $y \in B(z^{j_0}, \epsilon)$.

Finally, we prove that $\{A_\lambda : \lambda \in c_0^+\}$ is a fundamental system of compact sets of $c_0\left(\bigoplus_{i=1}^\infty \ell_p^i\right)$. Let $K \subset c_0\left(\bigoplus_{i=1}^\infty \ell_p^i\right)$ a compact subset. For each $m \in \mathbb{N}$ we define $\lambda_m = \sup_{z \in K}\left(\sum_{i \in I(m)} |z_i|^p\right)^{1/p}$. We claim that $\lambda = (\lambda_m)_m \in c_0^+$. Indeed, given $\epsilon > 0$, there are $z^1, ..., z^l \in K$ such that $K \subset \bigcup_{j=1}^l B(z^j, \frac{\epsilon}{2})$. As $z^j \in c_0\left(\bigoplus_{i=1}^\infty \ell_p^i\right)$, then for each $j = 1, ..., l$, there is $N_j \in \mathbb{N}$ such that $\left(\sum_{i \in I(m)} |z_i^j|^p\right)^{1/p} \leq \frac{\epsilon}{2}, \quad \text{for all} \quad m \geq N_j$. If $N = \max\limits_{1 \leq j \leq l} N_j$, then $\left(\sum_{i \in I(m)} |z_i^j|^p\right)^{1/p} \leq \frac{\epsilon}{2}, \quad \text{for all} \quad m \geq N \quad \text{e} \quad 1 \leq j \leq l$.

Let $z \in K$, so there is $1 \leq j_0 \leq l$, such that $\|z - z^{j_0}\| \leq \frac{\epsilon}{2}$. That means,

$$\left(\sum_{i \in I(m)} |z_i|^p\right)^{1/p} - \left(\sum_{i \in I(m)} |z_i^{j_0}|^p\right)^{1/p} \leq \left(\sum_{i \in I(m)} |z_i - z_i^{j_0}|^p\right)^{1/p} \leq \frac{\epsilon}{2},$$

for all $m \in \mathbb{N}$. So $\left(\sum_{i \in I(m)} |z_i|^p\right)^{1/p} \leq \epsilon \quad \text{for all} \quad m \geq N$. That is , for all $m \geq N$ $\sup_{z \in K}\left(\sum_{i \in I(m)} |z_i|^p\right)^{1/p} \leq \epsilon$. As a consequence, $|\lambda_m| = \sup_{z \in K}\left(\sum_{i \in I(m)} |z_i|^p\right)^{1/p} \leq \epsilon$, for all $m \geq N$, and $\lambda \in c_0^+$. If $w = (w_i)_i \in K$, we have $m \in \mathbb{N}$ $\left(\sum_{i \in I(m)} |w_i|^p\right)^{1/p} \leq \sup_{z \in K}\left(\sum_{i \in I(m)} |z_i|^p\right)^{1/p} = \lambda_m$. So $w \in A_\lambda$, and we proved that $\{A_\lambda : \lambda \in c_0^+\}$ is a fundamental system of compact sets of $c_0\left(\bigoplus_{i=1}^\infty \ell_p^i\right)$. ∎

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BANACH LATTICES OF LINEAR OPERATORS AND OF HOMOGENEOUS POLYNOMIALS NOT CONTAINING C_0

G. BOTELHO[1], V. C. C. MIRANDA[2] & P. RUEDA[3]

[1]Universidade Federal de Uberlândia, IME, MG, Brazil, botelho@ufu.br
[2]Universidad Federal do ABC, CMCC, SP, Brazil, colferaiv@gmail.com
[3]Universidad de Valencia, Spain, pilar.rueda@uv.es

Abstract

In this talk, we will present the results contained in Section 4 of the preprint [1]. In particular, we study when the following are equivalent for $\mathcal{I} = \mathcal{K}$ or $\mathcal{I} = \mathcal{W}$: (1) The space $\mathcal{P}^r(^nE;F)$ of regular polynomials contains no copy of c_0. (2) $\mathcal{P}^r_{\mathcal{I}}(^nE;F)$ contains no copy of c_0. (3) $\mathcal{P}^r_{\mathcal{I}}(^nE;F)$ is a projection band in $\mathcal{P}^r(^nE;F)$. (4) Every positive polynomial in $\mathcal{P}^r(^nE;F)$ belongs to $\mathcal{P}^r_{\mathcal{I}}(^nE;F)$. The result we obtain in the compact case can be regarded as a lattice polynomial Kalton theorem. Most of our results and examples are new even in the linear case $n = 1$.

1 Introduction

A classical problem in Functional Analysis consists in studying embeddability of c_0 in spaces of bounded linear operators between Banach spaces. One of the most known results in this direction is Kalton's theorem [2, Theorem 6] which states that for a Banach space X with an unconditional finite-dimensional expansion of the identity and an infinite dimensional Banach space Y, the space $\mathcal{L}(X;Y)$ of all bounded linear operators from X to Y contains no copy of c_0 if and only if every bounded linear operator from X to Y is compact. In [3], S. Pérez studied the embeddability of c_0 in the space $\mathcal{P}(^nX;Y)$ of all continuous n-homogeneous polynomials. In the lattice setting, this issue is specially important because the non embeddability of c_0 in a Banach lattice is equivalent to the lattice being a KB-space. In this direction, F. Xanthos [4] gave the following version of Kalton's theorem for the Banach lattice $\mathcal{L}^r(^nE;F)$ of all regular linear operators: for an atomic Banach lattice E with order continuous norm and an arbitrary Banach lattice F, $\mathcal{L}^r(^nE;F)$ contains no copy of c_0 if and only if every positive linear operator from E to F is compact (see [4, Theorem 2.9]).

The interest in studying polynomial versions of well known results or properties in Banach lattice theory have been considerably increased recently. It is then a natural question to seek for a "lattice polynomial version" of Kalton's theorem [2, Theorem 6]. The main purpose of this manuscript is to obtain conditions on the Banach lattice E and F so that the Banach lattice $\mathcal{P}^r(^nE;F)$ of all regular n-homogeneous polynomials from E to F contains no copy of c_0 if and only if every positive n-homogeneous polynomial from E to F is compact. In order to achieve this result, a complete lattice norm on the space $\mathcal{P}^r_{\mathcal{K}}(^nE;F)$, which is the linear span of all positive compact n-homogeneous polynomials from E to F, is introduced in a more general way:

Theorem 1.1. *Let E and F be Banach lattices with F Dedekind complete. Suppose that $\mathcal{A}$ is a closed subspace of $\mathcal{P}(^nE;F)$ such that (E,F) satisfies the $\mathcal{A}$-domination property, that is for all positive n-homogeneous polynomials $P,Q\colon E\to F$ with $0\le P\le Q\in\mathcal{A}$, it holds $P\in\mathcal{A}$. Thus*

$$\|P\|_{\mathcal{A},r} := \inf\left\{\|Q\| : Q\in\mathcal{A}^+, Q\ge |P|\right\}$$

defines a complete lattice norm on $\mathcal{A}^r = span\{\mathcal{A}^+\}$, that is, $(\mathcal{A}^r, \|\cdot\|_{\mathcal{A},r})$ is a Banach lattice. Moreover, $\|P\|_{\mathcal{A},r} = \|P\|_r$ for every $P\in\mathcal{A}^r$ and $\mathcal{A}^r$ is an ideal in $\mathcal{P}^r(^nE;F)$.

Considering $\mathcal{A} = \mathcal{K}$ or $\mathcal{A} = \mathcal{W}$, we get the two following interesting examples:

Examples: (a) If E is a Banach lattice and F is an atomic Banach lattice with order continuous norm, we obtain from Theorem 1.1 for $\mathcal{A} = \mathcal{P}_{\mathcal{K}}(^nE;F)$ that $(\mathcal{P}^r_{\mathcal{K}}(^nE;F), \|\cdot\|_{\mathcal{K},r})$ is a Banach lattice such that $\|P\|_{\mathcal{K},r} = \|P\|_r$ for every $P \in \mathcal{P}^r_{\mathcal{K}}(^nE;F)$.

(b) If E is a Banach lattice and F is a Banach lattice with order continuous norm, we obtain from Theorem 1.1 for $\mathcal{A} = \mathcal{P}_{\mathcal{W}}(^nE;F)$ that $(\mathcal{P}^r_{\mathcal{W}}(^nE;F), \|\cdot\|_{\mathcal{W},r})$ is a Banach lattice such that $\|P\|_{\mathcal{W},r} = \|P\|_r$ for every $P \in \mathcal{P}^r_{\mathcal{W}}(^nE;F)$.

2 Main Results

In this Section we will enunciated the two main results proved in [1, Section 4].

Theorem 2.1. *If E is a Banach lattice that fails the dual positive Schur property and F is an infinite dimensional atomic Banach lattice with order continuous norm, then the following are equivalent for every $n \in \mathbb{N}$:*
(1) *$(\mathcal{P}^r(^nE;F), \|\cdot\|_r)$ contains no copy of c_0.*
(2) *$(\mathcal{P}^r_{\mathcal{K}}(^nE;F), \|\cdot\|_{\mathcal{K},r})$ contains no copy of c_0.*
(3) *$\mathcal{P}^r_{\mathcal{K}}(^nE;F)$ is a projection band in $\mathcal{P}^r(^nE;F)$.*
(4) *Every positive n-homogeneous polynomial from E to F is compact.*

It is important noticing that Theorem 2.1 above is the lattice polynomial version of the famous Kalton's theorem. As a corollary, we have:

Corollary 2.1. *If F is an infinite dimensional atomic Banach lattice with order continuous norm and every positive n-homogeneous polynomial $P\colon E \to F$ is compact, then $\mathcal{P}^r(^nE;F)$ has order continuous norm. In addition, if E fails the dual positive Schur property, then $\mathcal{P}^r(^nE;F)$ is a KB-space.*

For weakly compact polynomials, we have the following:

Theorem 2.2. *Let $n \in \mathbb{N}$, let E be a Banach lattice that fails the positive Grothendieck property and let F be an atomic Dedekind complete Banach lattice such that $\mathcal{P}^r(^nE;F)$ has order continuous norm. Then, the following are equivalent:*
(1) *$(\mathcal{P}^r(^nE;F), \|\cdot\|_r)$ contains no copy of c_0.*
(2) *$(\mathcal{P}^r_{\mathcal{W}}(^nE;F), \|\cdot\|_{\mathcal{W},r})$ contains no copy of c_0.*
(3) *$\mathcal{P}^r_{\mathcal{W}}(^nE;F)$ is a projection band in $\mathcal{P}^r(^nE;F)$.*
(4) *Every positive n-homogeneous polynomial from E to F is weakly compact.*

An application of Theorem 2.2 is provided as well:

Corollary 2.2. *Let $n \in \mathbb{N}$ and let E, F be two Banach lattices with F Dedekind complete such that $\mathcal{P}^r(^nE;F)$ has order continuous norm. If E fails the positive Grothendieck property, F is atomic and every positive n-homogeneous polynomial $P\colon E \to F$ is weakly compact, then $\mathcal{P}^r(^nE;F)$ is a KB-space.*

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LAGRANGIAN SOLUTIONS OF WAVELIKE VECTOR FIELDS AND APPLICATIONS TO VLASOV-MAXWELL SYSTEM

H. BORRIN[1] & M. SANTOS[1]

[1]Unicamp, IMECC, SP, Brazil,
h216763@dac.unicamp.br, msantos@ime.unicamp.br

Abstract

In this proposal, we shall present a construction of solutions for the transport and continuity equations similar to the characteristics method–known as Lagrangian approach, first introduced by Crippa-De Lellis [1]–for vector fields which can be written as a "retarded convolution" of a singular kernel and a L^p function. The rough nature of such vector fields necessitate finer estimates involving the composition of maximal operators and singular kernels, and so the work of DiPerna-Lions [2] is not applicable. As an application, we give conditions on solutions to Vlasov-Maxwell system so that weak, renormalized, and Lagrangian solutions are all equivalent. In particular, it gives an explicit formula for its solutions depending on the associated flow.

1 Introduction

The classical transport (if $\eta = 1$) and continuity (if $\eta = 0$) equations

$$\begin{cases} \partial_t u + \operatorname{div}(\boldsymbol{b}u) = \eta u \operatorname{div} \boldsymbol{b} & \text{in} \quad [0,\infty) \times \mathbb{R}^d; \\ u_{t=0} = u_0 & \text{on} \quad \mathbb{R}^d \end{cases} \tag{1}$$

are maybe the simplest partial differential equations with major theory developments in the last forty years. Before the seminal work of DiPerna-Lions [2], the best widely known result concerning well-posedness of (1) was due to Osgood, which in turn was an improvement on the classical Cauchy-Lipschitz theory. The constructed solutions were via the characteristics method: by establishing well-posedness on the flow equation for a fixed $x \in \mathbb{R}^d$

$$\begin{cases} \partial_t \boldsymbol{X}(t,s,x) = \boldsymbol{b}_t(\boldsymbol{X}(t,s,x)) & \text{in} \quad [0,\infty); \\ \boldsymbol{X}(s,s,x) = x, \end{cases} \tag{2}$$

one may construct an unique solution explicitly depending on initial data, $\boldsymbol{X}$, and $\operatorname{div} \boldsymbol{b}$. The gap between vector fields satisfying the Osgood condition and in Sobolev spaces $W^{1,p}_{\text{loc}}$–and more generally BV vector fields–was filled by DiPerna-Lions [2] and Ambrosio [3]. The theory does not rely on solving in the ODE level (2), but rather on a special structure on (1): for u solutions of transport equation, a composition $\beta \circ u$ is also a solution for any $\beta \in C^1$. As a byproduct, one has the well-posedness of (2) in a renormalized sense. This approach–known as renormalization technique–does not extend for much larger vector fields spaces, as proven by the striking example by Depauw [4]. The new approach developed by Crippa-De Lellis [1]–known as Lagrangian approach–is an adaptation of the characteristic method, in a sense that it builds solutions of (1) from (2); this is known as a Lagrangian solution. There were many extensions of the technique, culminating on the well posedness of (2) for vector fields written locally as a sum of convolutions of singular kernels and BV functions; see [5].

We shall present a new generalization, replacing the convolution with a "retarded convolution", in the sense that

$$\boldsymbol{b}^j_t(x) = \sum_{k=1}^{m} \int_{B_t} K^{jk}(y) g^k_{t-|y|}(x-y) \mathrm{d}y \tag{3}$$

for K^{jk} singular kernels and g^k summable functions. The main motivation is the Vlasov-Maxwell system, where the vector field is written in terms of the physical velocity and electromagnetic fields–the latter solving a non homogeneous wave equation in $\mathbb{R}^3$. The techniques heavily parallel the quasistatic approximations of Maxwell equations in nonrelativistic and relativistic cases by Ambrosio-Colombo-Figalli [6] and Borrin-Marcon [3].

2 Main Results

The main result is threefold: firstly, we establish the well-posedness of (2) for vector fields with structure $\boldsymbol{b}_t(x) = (\Gamma\chi_{B_r}) * g_t(x)$, where Γ is a singular kernel and χ_A is an indicator function of a set A, extending the results of [5]; secondly, as a byproduct of the aforementioned result, we obtain the well-posedness of (2) for vector fields which can be written as (3). More precisely, the following holds:

Theorem 2.1. *Let $\boldsymbol{b}$ be a vector field written as $\boldsymbol{b}_t(x) = (\Gamma\chi_{B_r}) * g_t(x)$, where g is a summable function, Γ is a singular kernel as in Calderón-Zygmund theory, satisfying $(1+|\cdot|)^{-1}\boldsymbol{b} \in L^1((0,T);(L^1+L^\infty)(\mathbb{R}^d))$, and $\operatorname{div}\boldsymbol{b} \in L^1((0,T);L^\infty(\mathbb{R}^d))$. Then there exists an unique solution of (2). Moreover, there exists a Lagrangian solution of transport and continuity equation (1). The thesis holds for vector fields (3) for K^{jk} kernels whose derivatives are singular à la Calderón-Zygmund and $g^k \in W^{1,1}((0,T);L^1(\mathbb{R}^d))$.*

Thirdly, we provide a condition for solutions of Vlasov-Maxwell system to be Lagrangian ones.

Theorem 2.2. *Let $\boldsymbol{b}_t(x,v) = (\xi(v), E_t(x) + \xi(v) \times H_t(x))$ for all $(x,v) \in \mathbb{R}^6$ and $t \in (0,T)$, where*

$$\begin{aligned}(\partial_{tt} - \Delta)E = -\nabla\rho + \partial_t J, &\quad (\partial_{tt} - \Delta)H = \operatorname{curl} J,\\ E_{t=0} = E_0, &\quad H_{t=0} = H_0,\\ \partial_t E_{t=0} = \operatorname{curl} H_0 - J_0, &\quad \partial_t H_{t=0} = -\operatorname{curl} E_0\end{aligned}$$

for $\rho = \int_{\mathbb{R}^3} f(x,v)\mathrm{d}v$ and $J = \int_{\mathbb{R}^3} \xi(v) f(x,v)\mathrm{d}v$ and solutions f of (1) with the aforementioned vector field $\boldsymbol{b}$ and ξ is a Lipschitz function. Then for compatible initial data and if $\partial_{tt}J$ is a summable function, we have that weak, renormalized and Lagrangian solutions are all equivalent.

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SHARP REGULARITY ESTIMATES FOR DEGENERATE EVOLUTION PROBLEMS IN ORLICZ SPACES

J. G. ARAÚJO[1] & M. D. S. AMARAL[2]

[1]Universidade Federal Rural do Semi-Árido, ATI, RN, Brazil, janielly@ufersa.edu.br
[2]Universidade da Integração Internacional da Lusofonia Afro-Brasileira, ICEN, Brazil, marceloamaral@unilab.edu.br

Abstract

We obtain sharp Hölder regularity for bounded weak solutions $u \in W^{1,G}$ with $G \in \Upsilon_{g_0}^{g_1}$ of generalized p-Laplacian type parabolic equations of the form

$$u_t - \operatorname{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f,$$

where $1 < g_0 \le g_1$ and $f \in L^{G,r}$ with $G \in \Upsilon_{\overline{g_0}}^{\overline{g_1}}$. We show the precise sharp expression of the exponent depending only on the universal parameters of the problem like g_0, $\overline{g_0}$, r and the dimension n.

1 Introduction

We deliver sharp regularity estimates for locally bounded solutions of the degenerate generalized p-Laplacian equation

$$u_t - \operatorname{div}\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f \tag{1}$$

in $Q_1 := (-1, 0] \times B_1(0)$. We assume the following growth condition (Lieberman [6]): there exist constants g_0, g_1 such that

$$0 < g_0 \le \frac{t \cdot g'(t)}{g(t)} \le g_1 \quad \forall\, t > 0 \tag{2}$$

and

$$G'(t) = g(t), \quad \text{with} \;\; g \in C^0([0, +\infty]) \cap C^1((0, +\infty]). \tag{3}$$

The number of N-functions that satisfy the conditions (2) and (3) is quite expressive. For example, if $g(t) = t^{p-1}$ with $g_0 = g_1 = p - 1$, we obtain the prototype of p-Laplacian. There are also interesting and different examples $g(t) = t^{\beta}\ln(\gamma t + \eta)$, with $\beta, \gamma, \eta > 0$ and $g_0 = \beta$ and $g_1 = \beta + 1$ or by discontinuous power transitions like

$$g(t) = \begin{cases} c_1 t^{\beta}, & \text{if} \quad 0 \le t \le t_0 \\ c_2 t^{\gamma} + c_3, & \text{if} \quad t \ge t_0 \end{cases}$$

where β, γ, t_0 are positive numbers, and c_1, c_2, c_3 are real numbers such that $g \in C^1([0, \infty))$ with $g_0 = \min(\beta, \gamma)$ and $g_1 = \max(\beta, \gamma)$, among others. This class of nonlinear evolution equations appear in many relevant applications of physics [10], fluid dynamics [9] and image processing .

The general nature of the g function implies that certain techniques used to solve problems involving the p-Laplace equation cannot be directly applied to equation (1). Indeed, some properties of power functions, such as $(st)^p = s^p t^p$, are no longer applicable to the function g and there is lack of any type of homogeneity of the g function, for example. Consequently, additional efforts are needed making the problem more intriguing and challenging.

2 Main Results

The main result is stated below. We use compactness and intrinsic scaling methods to prove it.

Theorem 2.1. *A locally bounded weak solution $u \in W^{1,G}$ of (1) with $G \in \Upsilon_{g_0}^{g_1}$ and $f \in L^{G,r}$ with $G \in \Upsilon_{\overline{g_0}}^{\overline{g_1}}$, satisfying*

$$\frac{1}{r} + \frac{n}{(\overline{g_0}+1)(g_0+1)} < 1 < \frac{2}{r} + \frac{n}{(\overline{g_0}+1)} \tag{1}$$

is locally Holder continuous with exponents

$$\alpha = \frac{[(g_0+1)(\overline{g_0}+1)-n]r-(g_0+1)(\overline{g_0}+1)}{(\overline{g_0}+1)\left[g_0 r-(g_0-1)\right]} \tag{2}$$

in space and α/θ in time, where $\theta := (1+\alpha) - \log_\rho(g(\rho^{(\alpha-1)}))$.

For instance, our result generalizes the sharp regularity exponent α provided in [8] in which was addressed the inhomogeneous p-laplace parabolic equation

$$u_t - div(|\nabla u|^{p-2}\nabla u) = f \in L^{q,r} \tag{3}$$

with $p \geq 2$ and obtained

$$\alpha = \frac{(pq-n)r-pq}{q\left[(p-1)r-(p-2)\right]}.$$

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ERROR ANALYSIS OF THE OBERBECK-BOUSSINESQ MODEL USING NITE ELEMENT APPROXIMATION FOR SPATIAL DISCRETIZATION

J. BRAVO-OLIVARES

Universidad del Bío-Bío, Chillán, Chile, jbravo.olivares@gmail.com

Abstract

We prove local error estimates for spatial discretization of the solution for the Oberbeck-Boussinesq approximation using conforming or non-conforming finite element. Second-order error estimates are obtained for the velocity, temperature and concentration, without compatibility conditions on the data and only imposing regularity condition on the velocity. This result is proved using energy methods based on sharp a priori estimates of the Oberbeck-Boussinesq approximation.

1 Introduction

Certain flows can become quite complex when temperature and concentration differences interact simultaneously. These flows can be modelled using the Oberbeck-Boussinesq approximation which consists of the incompressible Navier-Stokes equations coupled with the heat and mass transfer equations.
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $n = 2$ or 3, and boundary $\partial\Omega$. The Oberbeck-Boussinesq approximation we consider, which describes the motion of a viscous-chemically-active fluid, is given by (see [2])

$$
\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p &= \boldsymbol{g}_1 + (\theta+\psi)\boldsymbol{g},\\
\frac{\partial \theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)\theta - \Delta\theta &= f - (\boldsymbol{u}\cdot\nabla)\theta_2,\\
\frac{\partial \psi}{\partial t} + (\boldsymbol{u}\cdot\nabla)\psi - \Delta\psi &= h - (\boldsymbol{u}\cdot\nabla)\psi_2,\\
\operatorname{div}\boldsymbol{u} &= 0.
\end{aligned}
\tag{1}
$$

Here, $\boldsymbol{u} = \boldsymbol{u}(x,t) \in \mathbb{R}^n$, $\theta = \theta(x,t) \in \mathbb{R}$, $\psi = \psi(x,t) \in \mathbb{R}$ and $p = p(x,t) \in \mathbb{R}$ represent the unknown velocity, temperature, concentration of material in the liquid and the pressure at the point $(x,t) \in \Omega \times (0,\infty)$, respectively. $\boldsymbol{g}(x,t)$,$\boldsymbol{j}(x,t)$, $f(x,t)$ and $h(x,t)$ are given source functions. $\boldsymbol{g}_1(x,t) = (\theta_2+\psi_2)\boldsymbol{g} + \boldsymbol{j}$ where θ_2 and ψ_2 are functions that result from a lifting process when the temperature and concentration are non-zero on the boundary (see [5, 4]). The fluid density and viscosity have been normalized.
Together with (1), we consider the following initial and boundary conditions

$$
\begin{aligned}
\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \quad \theta(x,0) = \theta_0(x), \quad \psi(x,0) = \psi_0(x),\\
\boldsymbol{u}(x,t) = 0, \quad \theta(x,t) = 0, \quad \psi(x,t) = 0,
\end{aligned}
\tag{2}
$$

where $\boldsymbol{u}_0, \theta_0$ and ψ_0 are given functions.
In this work, we deal with the error analysis of finite element solution of the Oberbeck-Boussinesq approximation. We prove second-order error estimates, using conforming or non-conforming elements, obtained without compatibility conditions on the data.
In order to obtain our error estimations, we need some assumption about Ω, appropriate regularity on the initial data, and the global existence of strong solution. These assumption we will be referred to as (A1), (A2) and (A3), respectively, which are similar to those assumed in [3] but extended for the steady Poisson problem, as well as

for temperature and concentration. We emphasize that assuming only (A3) assumption for the velocity $\boldsymbol{u}$, we can conclude the same regularity for the temperature θ and concentration ψ.

Additionally, we require certain assumptions regarding the finite element approximation. Consider the discrete velocity $\boldsymbol{u}_h$, temperature θ_h, concentration ψ_h and pressure p_h , which are determined in finite element spaces denoted by $\boldsymbol{H}_h$, M_h (for temperature and concentration) and L_h, respectively, where h is a parameter representing mesh size. These spaces are assumed to possess (at least) the typical properties for $\boldsymbol{H}_h$, M_h (see (B1)-(B4) and (B4') properties in [3] for $\boldsymbol{H}_h$ of which we assume (B2)-(B3) hold for M_h), consisting of piecewise linear functions, while L_h consists of piecewise constant functions.

2 Main Results

Our main result is summarized in the following theorem

Theorem 2.1. *Let Ω be a convex polygon or polyhedron and suppose the conditions (A1), (A2), (A3) and (B1), (B2), (B3), (B4) are satisfied. Further suppose the discrete initial velocity, temperature and concentration $\boldsymbol{u}_{h_0} \in \boldsymbol{V}_h$, $\theta_{h_0} \in M_h$ and $\psi_{h_0} \in M_h$ are chosen to satisfy*

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_{h_0}\| \le h^2 M_4, \quad \|\theta_0 - \theta_{h_0}\| \le h^2 M_4, \quad \|\psi_0 - \psi_{h_0}\| \le h^2 M_4, \tag{1}$$

for some constant M_4. Then, the solution $(\boldsymbol{u}_h(t), \theta_h(t), \psi_h(t))$ associated to the discrete problem (1) satisfies

$$\|(\boldsymbol{u} - \boldsymbol{u}_h)(t)\| \le h^2 C_1(t), \quad \|(\theta - \theta_h)(t)\| \le h^2 C_1(t), \quad \|(\psi - \psi_h)(t)\| \le h^2 C_1(t), \tag{2}$$

with $0 \le t < T$, where $C_1(t)$ is a continuous function of $t \in [0, T)$. In addition to t, $C_1(t)$ depends only on the domain Ω, the constants $\kappa_1, \cdots \kappa_6$, M_4 appearing in assumptions (B1), (B3), (B4) and (1), and bounds M_1, M_2, M_3 appearing in assumptions (A2) and (A3).

The time interval $[0, T)$ is the same as in assumption (A3). If condition (B4') is satisfied, in addition to the previous assumptions, then any solution $p_h(t)$ associated to the discrete problem of (1) satisfies

$$\|(p - p_h)(t))\|_{L^2/\mathbb{R}} \le h C_2(t) \tag{3}$$

with $0 < t < T$, where $C_2(t)$ is a continuous function of t in the open interval $(0, T)$. $C_2(t)$ depends on the constants κ_0 appearing in (B4'), as well as the quantities determining $C_1(t)$.

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SUSPENSION BRIDGE WITH KELVIN-VOIGT DAMPING

CARLOS RAPOSO[1], LEANDRO CORREIA[2], JOILSON RIBEIRO[3] & LUIZ GUTEMBERG[1]

[1]Universidade Federal do Pará, UFPA, PA, Brazil, carlos.raposo@ufba.br, luizmiranda@ufpa.br
[2]Universidade Estadual do Sudoeste da Bahia, UESB, BA, Brazil, leandro.araujo@uesb.edu.br
[3]Universidade Federal da Bahia, UFBA, BA, Brazil, joilsonor@ufba.br
1

Abstract

This work deals with a suspension bridge model with Kelvin-Voigt damping. We use semigroup theory proving the existence of solution applying the Lumer-Phillips theorem. Moreover, we obtain exponential stability of the semigroup associated with the energy space.

1 Introduction

In this work we study the existence of solutions and analyticity for the initial boundary value problem of a suspension bridge with a Kelvin-Voigt viscoelastic damping

$$u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) - \gamma_1 u_{txx} = 0 \quad \text{em } (0,L)\times(0,\infty) \tag{1}$$

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x + \lambda(\varphi - u) - \gamma_2\varphi_{txx} = 0 \text{ em } (0,L)\times(0,\infty) \tag{2}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) - \gamma_3\psi_{txx} = 0 \quad \text{em } (0,L)\times(0,\infty) \tag{3}$$

The equations above are considering that the deck has negligible transversal section dimensions compared to the length (span of the bridge), it is modeled in Timoshenko's theory as a one-dimensional extensible beam of length L, see [6]. As in [2], where we are denoting by $\varphi = \varphi(x,t)$ the displacement of the cross-section on the point $x \in (0,L)$, by $\psi = \psi(x,t)$ the rotation angle of the cross-section and the suspender cables are assumed to be linear elastic springs with standard stiffness $\lambda > 0$. The constant $\alpha > 0$ is the elastic modulus of the string (holding the main cable to the deck). The positive coefficients ρ_1 and ρ_2 are the mass density and the moment of mass inertia of the beam, respectively. Moreover, b represents the cross section's rigidity coefficient, and k represents the elasticity's shear modulus. Finally, the constants $\gamma_1 > 0$ and $\gamma_2, \gamma_3 \geq 0$ are the coefficients of the damping force.

System (1.1) – (1.3) is subject to initial data and Dirichlet boundary conditions

$$\begin{cases} u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x),\, x \in (0,L), \\ \varphi(x,0) = \varphi_0(x), \;\; \varphi_t(x,0) = \varphi_1(x),\, x \in (0,L), \\ \psi(x,0) = \psi_0(x), \;\; \psi_t(x,0) = \psi_1(x),\, x \in (0,L), \end{cases} \tag{4}$$

$$\begin{cases} u(0,t) = u(L,t) = 0,\, t \geq 0, \\ \varphi(0,t) = \varphi(L,t) = 0,\, t \geq 0, \\ \psi(0,t) = \psi(L,t) = 0,\, t \geq 0. \end{cases} \tag{5}$$

We introduce the Hibert Space

$$\mathcal{H} = H_0^1(0,L)\times L^2(0,L)\times H_0^1(0,L)\times L^2(0,L)\times H_0^1(0,L)\times L^2(0,L)$$

endowed with the following inner product,

$$\langle U, \tilde{U}\rangle_{\mathcal{H}} = \int_0^L v\overline{\tilde{v}}dx + \alpha\int_0^L u_x\overline{\tilde{u}}_x dx + \rho_1\int_0^L w\overline{\tilde{w}}dx + \rho_2\int_0^L z\overline{\tilde{z}}dx + b\int_0^L \psi_x\overline{\tilde{\psi}}_x$$
$$+ \lambda\int_0^L (\varphi - u)(\overline{\tilde{\varphi}} - \overline{\tilde{u}})dx + k\int_0^L (\varphi_x + \psi)(\overline{\tilde{\varphi}}_x + \overline{\tilde{\psi}})dx,$$

being $U = (u, v, \varphi, w, \psi, z)^T$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{w}, \tilde{\psi}, \tilde{z})^T$, with $u_t = v, \varphi_t = w$ and $\psi_t = z$. With this notation, we rewrite $(1.1) - (1.3)$ as the following first-order Cauchy problem

$$\begin{cases} U_t - \mathcal{A}U = 0, \\ U(0) = U_0, \end{cases} \tag{6}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H},$ with $D(\mathcal{A}) = [H_0^1(0, L) \cap H^2(0, L) \times H_0^1(0, L)]^3$ is defined by (6).

2 Main Results

For existence of solution, the main idea is to use the well-known Lummer-Phillips Theorem (see [3]). As $D(\mathcal{A})$ is dense in $\mathcal{H}$, to get that $\mathcal{A}$ is the infinitesimal generator of $S(t) = e^{\mathcal{A}t}$, a C_0-semigroup of contractions on $\mathcal{H}$, we prove that $\mathcal{A}$ is dissipative and that $0 \in \rho(\mathcal{A})$ the resolvent set of $\mathcal{A}$, obtaining the following theorem.

Theorem 2.1. *Let $U_0 \in \mathcal{H}$, then there exists a unique weak solution U of problem (6) satisfying $U \in C^0([0, +\infty); \mathcal{H})$. Moreover, if $U_0 \in D(\mathcal{A})$, then $U \in C^0([0, +\infty); D(\mathcal{A})) \cap C^1([0, +\infty); \mathcal{H})$.*

For the assimptotic behavior, as consequence of Gearhart-PrÃ¼ss Theorem (see [1]), we obtain the main result as follow.

Theorem 2.2. *The C_0-semigroup of contractions $S(t) = e^{\mathcal{A}t}, t \geq 0$, generated by $\mathcal{A}$ is exponentially stable.*

Proof. By contradiction, exists θ and a sequence $\beta^n \to \theta, |\beta^n| < |\theta|$, with $||(i\beta^n - A)^{-1}||_{L(\mathcal{H})} \to \infty$ and for all $M > 0$, there is a $n_0 \in \mathbb{N}$ such that $n > n_0$, then $||(i\beta^n - A)^{-1}||_{L(\mathcal{H})} > M$.

Using the PoincarÃ© and Gagliardo-Niremberg inequalitys, we obtain that $i\mathbb{R} \in \rho(A)$ and, with a more careful calculation, $\overline{\lim\limits_{|\beta| \to \infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$, and the result follows from Gearhart-PrÃ¼ss Theorem.

□

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EXISTENCE OF A LOCAL SOLUTION FOR A THERMOELASTIC PLATE MODEL WITH AN UNBOUNDED DOMAIN

L. J. S. DOS ANJOS[1] & C. BURIOL[2]

[1]Universidade Regional do Noroeste do Estado do Rio Grande do Sul , UNIJUÍ, RS, Brazil, luis.anjos@sou.unijui.edu.br
[2]Universidade Federal de Santa Maria, UFSM, RS, Brazil, celene.buriol@ufsm.br

Abstract

This work aims to establish the local existence of a solution to a system of hyperbolic partial differential equations, specifically a formulation of the heat equation modeled by Cattaneo. We use the theory of semigroups to approach the solution of this system taking advantage of Functional Analysis techniques, in particular, we use the Banach Fixed Point Theorem in a Hilbert space.

1 Introduction

In this work we consider three scalar functions $u(x,t), q(x,t)$ and $\theta(x,t)$ satifying the coupled system

$$\begin{cases} u_{tt} - \mu u_{xxtt} + u_{xxxx} + \alpha u - M\left(\int_{\mathbb{R}} u_x^2 dx\right) u_{xx} + \delta\theta_{xx} = 0;\ t > 0, x \in \mathbb{R}, \\ \theta_t + kq_x - \delta u_{xxt} = 0; \quad t > 0,\ x \in \mathbb{R}, \\ \tau q_t + q + k\theta_x = 0; \quad t > 0,\ x \in \mathbb{R}. \end{cases} \tag{1}$$

in $\Omega = (0, L)$ with initial conditions

$$u(x,0) = u_0(x); u_t(x,0) = u_1(x); \theta(x,0) = \theta_0(x); q(x,0) = q_0(x) \tag{2}$$

and boundary conditions

$$u(0,t) = u(L,t) = u_{xx}(0,t) = u_{xx}(L,t) = \theta(0,t) = \theta(L,t) = 0, \tag{3}$$

such that μ, δ, k and τ are positive constants experimentally provided and $u := u(x,t)$ com $x \in \mathbb{R}$ e $t \in [0, \infty]$. Furthermore, the constants we are considering in 1 are usually associated with the following: τ is the "relaxation" time, δ is a coupling constant for 1 and θ and q denotes the difference to a fixed temperature.

The total energy is associated to 1 is

$$E(t) = \int_0^L \left(u_t^2 + \mu u_{xt}^2 + u_{xx}^2 + \theta^2 + \tau q^2\right) dx + \widehat{M}\left(\int_0^L u_x^2 dx\right) \tag{4}$$

where $\widehat{M}(\lambda) = \displaystyle\int_0^\lambda M(s)ds$ for all $s \geq 0$ with $M(s) \geq 0$ is a $C^1(\Omega)$ real function.
The model (1) describes thermoelastic deformations of a linear plate equation under the presence of thermal effects modeled by Cattaneo's Law (see [1] and [3]). Our main result establishes that there is a local solution for the model

2 Main Results

Theorem 2.1. *(Local solution) Let A and F be two functions in the hilbert space then if* $\{u_0, u_1, \theta_0, q_0\} \in \mathbf{H^4} \times \mathbf{H^3} \times \mathbf{H^2} \times \mathbf{H^2}$ *there is a unique* $u(t) = \{u(t); u_t(t); \theta(t); q(t)\}$ *e* T_{max} *such that* $U \in C([0, T_{max}); \ \mathbf{H^4} \times \mathbf{H^3} \times \mathbf{H^2} \times \mathbf{H^2}) \ \cap \ C^1([0, T_{max}]; \mathbf{H^2} \times \mathbf{H^1} \times \mathbf{L^2} \times \mathbf{L^2})$.

Proof. First, we reformulate the original problem as an abstract first-order differential equation in a Hilbert space. Let $\mathbf{U}(t) = \{u(t), u_t(t), \theta(t), q(t)\}$ and define an operator $\mathcal{A}$ such that the original system can be written as

$$\frac{d\mathbf{U}}{dt} = \mathcal{A}\mathbf{U} + \mathcal{F}(\mathbf{U}),$$

where $\mathcal{A}$ represents the linear part and $\mathcal{F}$ represents the nonlinear part.

Now, we apply Picard iteration to construct a sequence of approximate solutions $\{\mathbf{U}_n\}$. Start with an initial value $\mathbf{U}_0$ and define the sequence by

$$\mathbf{U}_{n+1}(t) = \mathbf{U}_0 + \int_0^t \left(\mathcal{A}\mathbf{U}_n(s) + \mathcal{F}(\mathbf{U}_n(s))\right) ds.$$

We show that this sequence converges in the appropriate function space using the Banach fixed-point theorem (contraction mapping principle).

To prove uniqueness, assume there are two solutions $\mathbf{U}_1$ and $\mathbf{U}_2$ both satisfying the initial conditions and the differential equation. Consider the difference $\mathbf{V} = \mathbf{U}_1 - \mathbf{U}_2$ and show that $\mathbf{V}$ satisfies a homogeneous equation with zero initial data:

$$\frac{d\mathbf{V}}{dt} = \mathcal{A}\mathbf{V} + (\mathcal{F}(\mathbf{U}_1) - \mathcal{F}(\mathbf{U}_2)).$$

Using Gronwall's inequality, we show that $\mathbf{V}(t) = 0$ for all $t \in [0, T_{max})$, proving uniqueness.

Next, we verify that the solution $\mathbf{U}(t)$ belongs to the desired function spaces by using regularity results for the linear operator $\mathcal{A}$ and the properties of the nonlinear term $\mathcal{F}$. Specifically, we show that

$$\mathbf{U}(t) \in C([0, T_{max}); \mathbf{H}^4 \times \mathbf{H}^3 \times \mathbf{H}^2 \times \mathbf{H}^2) \cap C^1([0, T_{max}]; \mathbf{H}^2 \times \mathbf{H}^1 \times \mathbf{L}^2 \times \mathbf{L}^2).$$

Finally, we determine T_{max} by showing that it is the maximal existence time for the solution. We use a continuation argument to prove that if the solution can be extended beyond T_{max}, then T_{max} is not maximal, leading to a contradiction.

Thus, we have shown that there exists a unique solution $\mathbf{U}(t)$ on $[0, T_{max})$ with the desired regularity properties. □

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EXISTENCE OF SOLUTIONS FOR A LOGARITHMICALLY DAMPED WAVE EQUATION

L. G. LONGEN[1] & R. C. CHARÃO[1]

[1]Universidade Federal de Santa Catarina, CFM, SC, Brazil
luisglongen@gmail.com, ruycharao@gmail.com

Abstract

In this work, we investigate the critical exponent for the nonexistence of global solutions for the dissipative wave equation, based on the L_θ operator introduced in [1]. More precisely, we consider the following Cauchy problem

$$\begin{cases} u_{tt} - \Delta u + L_\theta u_t = |u|^p, & t \geq 0,\ x \in \mathbb{R}^n. \\ (u, u_t)(0, x) = (0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $\theta \in [0, \frac{1}{2})$ and $p > 1$, where the operator $L_\theta := D(L_\theta) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined as

$$(L_\theta f)(x) := \mathcal{F}^{-1}\left(\log(1 + |\xi|^{2\theta})\hat{f}(\xi)\right)(x).$$

To obtain the existence results, we utilize estimates to the decay of solutions to the linear associated problem. In this sense, we improve some previous results obtained in [1], expanding the range for which there are solutions to the linear problem from $\theta \in [0, \frac{5}{12})$ to $\theta \in [0, \frac{1}{2})$. The existence results in the supercritical case for the semilinear problem are obtained by applying a contraction principle.

1 Introduction

The main results of existence of global solutions for the supercritical case for problem (1) are obtained by appying Duhamel's Principle and Picard's Contraction Theorem. To achieve this, we must obtain sharp decay estimates for the linear associated problem to (1), that is,

$$\begin{cases} u_{tt} - \Delta u + L_\theta u_t = 0, \\ u(0, x) = 0, \quad u_t(0, x) = u_1(x). \end{cases} \tag{1}$$

The solution to the linear problem can be written as a convolution in the spatial variable, $\overline{u}(t, x) = K_1(t, x) * u_1(x)$, where K_1 is the second fundamental solution to (1). Then, applying Duhamel's Principle, a function $u \in \mathcal{C}\left(0, T), H^k(\mathbb{R}^n)\right)$, $k \geq 0$ is the unique global (weak) solution for (1) if, and only if, it satisfies

$$u(t, \cdot) = \overline{u}(t, \cdot) + \int_0^t K_1(t - s, x) * |u(s, x)|^p \, \mathrm{d}s \quad \text{in } H^k(\mathbb{R}^n). \tag{2}$$

The integral operator in the right-hand side of equation (2) is called G and defined for a certain adequate Banach space $X(T)$. Then to apply Picard's Contraction Principle, we must show that G maps balls into balls in $X(T)$ and that it is a contraction in $X(T)$. It becomes clear then that we need to estimate norms such as $|||D|^k|u|^p * K_1||_{L^2}$, and in order to do this we use the same decomposition as in [1],

$$\hat{f}\hat{K}_1 = \varphi_f + \sum_{j=1}^{6} F_j. \tag{3}$$

2 Main Results

The main results are the improved estimates for the linear problem and the existence results for the semilinear problem, which includes the determination of the critical exponent.

Proposition 2.1. *Let $k \in \mathbb{N}$. Assume that $n = 1$ and $\theta \in [0, \frac{1}{4}]$, or that $n \geq 2$ and $\theta \in [0, \frac{1}{2})$, $n \in \mathbb{N}$. If $f \in L^1(\mathbb{R}^n) \cap L^{1,2\theta}(\mathbb{R}^n)$, then*

$$\|\partial_x^k f(t,\cdot) * K_1(t,\cdot)\|_{L^2}^2 \lesssim \|f\|_{L^1 \cap L^{1,2\theta}}^2 \left((1+t)^{-\frac{n+2k-4\theta}{2\theta}} + \frac{1}{\theta}(1+t)^{-\frac{n+2k-4\theta}{2(1-\theta)}} \right), \qquad t \gg 1.$$

Proof Let $m \in \mathbb{N}$ such that $\theta \in \left[0, \frac{2m-1}{4m}\right)$, and consider the ball $B_\eta := \{\xi \in \mathbb{R}^n : |\xi| \leq \eta^m\}$, where η is a small fixed value such that the characteristic roots of the problem (1) are real for $|\xi| < \eta$. Using the decomposition given in (3), we estimate each of the norms of $|\xi|^k |F_j|$, $j = 1, ...6$, in L^2 and the L^2-norm of the asymptotic profile φ_f. Then, by taking the worst obtained decay rate, we get the result in the low-frequency zone $|\xi| < \eta^m$. For the mid-frequency $|\xi| \in (\eta^m, \delta)$ and high-frequency zone $|\xi| > \delta$, we can easily show that the decay rate is exponential. Combining the results in all three region, we obtain the desired result.

■

Theorem 2.1. *Assume that $n = 1$ and $\theta \in [0, \frac{1}{4}]$, or that $n = 2$ and $\theta \in [0, \frac{1}{2})$, $n \in \mathbb{N}$. Also, let $p > p_c$, with*

$$p_c := p_c(n, \theta) = 1 + \frac{2}{n - 2\theta}.$$

Then, there exists $\varepsilon > 0$ such that, for initial data $(0, u_1)$ satisfying $u_1 \in \mathcal{A} := L^1(\mathbb{R}^n) \cap L^{1,2\theta}(\mathbb{R}^n)$, with $\|u_1\|_{\mathcal{A}} \leq \varepsilon$, there exists a global solution to problem (1), $u \in \mathcal{C}\left([0, \infty), H^1(\mathbb{R}^n)\right)$.

Proof We set, for $T > 0$, the evolution space $X(T) := \mathcal{C}\left([0, T], H^1(\mathbb{R}^n)\right)$ equipped with the norm

$$\|v\|_{X(T)} := \sup_{t \in [0,T]} \left(\theta(1+t)^{\frac{n-4\theta}{4(1-\theta)}} \|v\|_{L^2} + \theta(1+t)^{\frac{n+2-4\theta}{4(1-\theta)}} \|\partial_x v\|_{L^2} \right).$$

Then, we apply Proposition (2.1) along with the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p} \lesssim \|u\|_{L^2}^{1-\alpha(p)} \|\partial_x u\|_{L^2}^{\alpha(p)}, \qquad \alpha(p) := n\left(\frac{1}{2} - \frac{1}{p}\right)$$

to estimate the norm of Gu in $X(T)$:

$$\|Gu\|_{L^2} \lesssim \frac{1}{\theta^3} \int_0^t (1+t-s)^{-\frac{n-4\theta}{4(1-\theta)}} (1+s)^{-\frac{n(p-1)-2\theta p}{2(1-\theta)}} \, ds \, \|u\|_{X(T)}^p \lesssim \frac{1}{\theta}(1+t)^{-\frac{n-4\theta}{4(1-\theta)}} \|u\|_{X(T)}^p, \tag{1}$$

$$\|\partial_x Gu\|_{L^2} \lesssim \frac{1}{\theta^3} \int_0^t (1+t-s)^{-\frac{n+2-4\theta}{4(1-\theta)}} (1+s)^{-\frac{n(p-1)-2\theta p}{2(1-\theta)}} \, ds \, \|u\|_{X(T)}^p \lesssim \frac{1}{\theta}(1+t)^{-\frac{n+2-4\theta}{4(1-\theta)}} \|u\|_{X(T)}^p. \tag{2}$$

The two above estimates imply that $G : X(T) \mapsto X(T)$. A similar computation shows that it is a contraction. Calculations (1) and (2) hold if the exponent $\frac{n(p-1)-2\theta p}{2(1-\theta)}$ is greater than 1, which is equivalent to

$$p > 1 + \frac{2}{n - 2\theta}.$$

This concludes our proof. ■

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STABILITY ANALYSIS OF A PARTIALLY DAMPED SUSPENSION BRIDGE BY FRICTION

L. G. R. MIRANDA[1], C. A. RAPOSO[1] & L. C. ARAUJO[2]

[1]Federal University of Pará, Salinópolis, PA, Brazil, luiz@ufpa.br, carlos.raposo@ufba.br
[2]Federal University of Bahia, Salvador-Bahia, Brazil

Abstract

This work considers a suspension bridge of length l where Timoshenko's theory models the deck. Using the semigroup Theory we analyze the existence and uniqueness of the solution and the asymptotic behavior of these solutions, more precisely, for the existence and uniqueness of the solution we use the Lumer-Phillips Theorem, and for the asymptotic behavior we use Gearhart-Herbst-Prüss-Huang Theorem and the result due to Borichev and Tomilov, to show exponential stability and polynomial stability with optimal stability rate, respectively.

1 Introduction

In 1943, Timoshenko published a work about suspension bridges, namely The Theory of Suspension Bridges, see [7, 8]. After that, in 1984, Hayashikawa and Watanabe used Hamilton's principle and Timoshenko's beam theory to study the Inoshima suspension Bridge (that connects Honshu and Shikoku in Japan) see [2].

We introduce a model of a suspension bridge, see Figure (1), given as a mechanical structure that carries vertical loads through the main cables modeled by an elastic string $u = u(x,t)$, which is coupled to the deck employing suspension cables, where x denotes the distance along the center line of the deck in its equilibrium configuration and t the time variable.

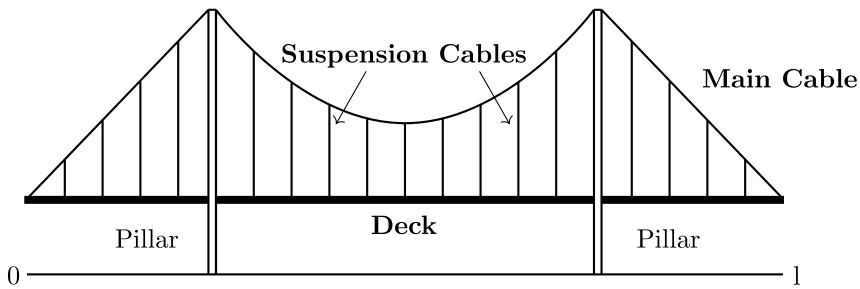

Figure 1. Suspension Bridge.

Considering that the deck has negligible transversal section dimensions compared to the length (span of the bridge), it is modeled in Timoshenko's theory [6]. Denoting by $\varphi = \varphi(x,t)$ the displacement of the cross-section on the point $x \in (0,l)$, by $\psi = \psi(x,t)$ the rotation angle of the cross-section, we have the following coupled system

$$u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \gamma_1 u_t = 0, \quad \text{in } (0,l) \times \mathbb{R}_+, \tag{1}$$

$$\rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 \varphi_t = 0, \quad \text{in } (0,l) \times \mathbb{R}_+, \tag{2}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma_3 \psi_t = 0, \quad \text{in } (0,l) \times \mathbb{R}_+, \tag{3}$$

subject to boundary conditions

$$u(0,t) = u(l,t) = \varphi(0,t) = \varphi(l,t) = \psi_x(0,t) = \psi_x(l,t) = 0, \quad t \geq 0. \tag{4}$$

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \varphi(x,0) = \varphi_0(x), \quad x \in (0,l), \tag{5}$$

$$\varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad x \in (0,l). \tag{6}$$

The suspender cables are assumed to be linear elastic springs with standard stiffness $\lambda > 0$. The constant $\alpha > 0$ is the elastic modulus of the string (holding the main cable to the deck). The positive coefficients ρ_1 and ρ_2 are the mass density and the moment of mass inertia of the beam, respectively. Moreover, b represents the cross section's rigidity coefficient, and κ represents the elasticity's shear modulus. Finally, γ_i $(i = 1, 2, 3)$, are non-negative parameters related to friction damping.

A suspension bridge with internal damping, where $\gamma_1, \gamma_2, \gamma_3 > 0$, was considered in [5]. Raposo et al. obtained that the solution not only decays exponentially but is also analytical.

2 Main Results

Our main results are the following theorems:

Theorem 2.1. *Suppose $\gamma_1 = 0$. Then the semigroup $S(t)$ associated with the system (1)–(6) is not exponentially stable independently of γ_1 and γ_2.*

Theorem 2.2. *Suppose $\gamma_1 > 0$ and $\gamma_2\gamma_3 = 0$ with one of them positive. Then if*

$$\chi_0 := \frac{\kappa}{\rho_1} - \frac{b}{\rho_2} = 0,$$

the semigroup $S(t)$ associated with the system (1)–(6) is exponentially stable. Otherwise, if $\chi_0 = 0$, it decays polynomially with optimal rate.

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CARLEMAN ESTIMATES FOR PARABOLIC EQUATIONS WITH SUPER STRONG DEGENERACY IN A SET OF POSITIVE MEASURE

B. S. V. ARAÚJO[1], R. DEMARQUE[2], J. C. O. FARIA[3] & L. VIANA[4]

[1] Universidade Federal de Campina Grande, Unidade Acadêmica de Matemática, Campina Grande, PB, Brazil, bsergio@mat.ufcg.edu.br

[2]]Universidad Federal Fluminense, Departamento de Ciências da Natureza, Rio das Ostras, RJ, Brazil, reginaldo@id.uff.br

[3]Universidade Estadual de Maringá, Departamento de Matemática, Maringá, PR, Brazil, jcofaria@uem.br

[4]Universidade Federal Fluminense, Departamento de Análise, Niterói, RJ, Brazil, luizviana@id.uff.br

Abstract

This work is concerned with the obtainment of new Carleman estimates for linear parabolic equations, where the second-order differential operator brings a super strong degeneracy in a positive measure subset of the spatial domain. In order to prove our main result, the control domain is supposed to contain the set of degeneracies. As a well-known consequence, we achieve a null controllability result in the current context.

1 Introduction

In this paper, we study the null controllability of the following degenerate parabolic of

$$u_t - (a(x)u_x)_x + c(x,t)u = f1_\omega \quad \text{in } Q := (0,1) \times (0,T), \tag{1}$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) = u_0(x), &\quad \text{in} \quad (0,1), \\ u(0,t) = u(1,t) = 0 &\quad \text{in} \quad (0,T), \end{aligned} \tag{2}$$

where $a \in W^{2,\infty}(0,1)$, $c \in L^\infty(Q)$, the control f belongs to $L^2(Q)$, $u_0 \in L^2(0,1)$, the control domain $\omega \subset (0,1)$ is a non-empty open interval and 1_ω denotes its associated characteristic function.

We say that the problem described by (1) and (2) is *null controllable* at time $T > 0$ if, for any $u_0 \in L^2(0,1)$, there exists a control function $f \in L^2(Q)$ such that the corresponding solution u satisfies

$$u(x,T) = 0 \quad \text{a. e. in } (0,1). \tag{3}$$

The null controllability of degenerate parabolic equations, such as (1) and (2), when the function $a = a(x)$ is weakly or strongly degenerate at $x = 0$, has been initially investigated in [2]. It is worth saying that, in that case, $a(x) = x^\alpha$, with $\alpha \in (0,1)$ for the weak case and $\alpha \in [1,2)$ for the strong one. However, the mentioned work also establishes that the *super strongly degenerate problem* ($\alpha \geq 2$) is not null controllable, in general.

Later, in [6], similar results were achieved when the degeneracy occurs in an interior point $x_0 \in (0,1)$. In this situation, it is supposed $a \in \mathcal{C}^1([0,1] - \{x_0\})$ satisfying $a(x_0) = 0$ and $a > 0$ in $[0,1] - \{x_0\}$. Additionally, it is assumed

(a) $\exists K \in (0,1)$ such that $(x - x_0)a'(x) \leq Ka(x)$ $\forall x \in [0,1] - \{x_0\}$ (for the weakly degenerate case);

(b) $a \in W^{1,\infty}(0,1)$ and $\exists K \in [1,2)$ such that $(x - x_0)a'(x) \leq Ka(x)$ $\forall x \in [0,1] - \{x_0\}$ (for the strongly degenerate case).

The function $a(x) = |x - x_0|^\alpha$, with $\alpha \in (0, 2)$, is a typical prototype for the investigation developed in [6], where the main results were achieved by assuming

$$x_0 \in \omega. \tag{4}$$

More recently, in [3], the results of [6] have been extended, by considering second-order operators that degenerate in an interval

$$[A, B] \subset \omega. \tag{5}$$

Of course, geometrical assumptions like (4) and (5) were not considered in [2], where it is explained the impossibility of having the null controllability property for the super strongly degenerate case ($\alpha \geq 2$), in general.

Encouraged by [1], this current work is a natural continuation of [3] and [6], for the super strongly degenerate problem, taking into consideration the hypotheses (5) and

$$\frac{1}{a} \notin L^1([0, A) \cup (B, 1]), a \in W^{2,\infty}(0, 1) \quad \text{and} \quad aa_{xx} \in W^{1,\infty}(0, 1). \tag{6}$$

We should say that the technical conditions given in (6) also appear in [4] and [5]. Roughly speaking, we have in mind a general function $a : [0, 1] \longrightarrow \mathbb{R}$, which behaves like

$$a(x) = (A - x)^\alpha 1_{[0,A)}(x) + (x - B)^\beta 1_{(B,1]}(x),$$

for each $x \in [0, 1]$, where $\alpha, \beta \geq 2$, and $1_{[0,A)}$ and $1_{(B,1]}$ denote characteristic functions.

2 Main Results

At this point, we are ready to present our main result:

Theorem 2.1. *Under the aforementioned hyphotheses on $a = a(x)$ and ω, solutions of the adjoint system associated with (1)-(2), in the super strongly degenerate escope, satisfy a Carleman type inequality. As a consequence, the system (1)-(2) is null controllable at any time $T > 0$, in the sense of (3).*

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HOW TO APPROXIMATE KIRCHHOFF EQUATION WITH DIRICHLET CONDICTION BY NONLOCAL DIFFUSION PROBLEMS WITH SMOOTH KERNELS

LUIZA CAMILE ROSA DA SILVA

Universidade de São Paulo, IME, Brazil, luiza@ime.usp.com

Abstract

In this work, we consider the nonlocal diffusion equation that involves three differents smooth kernels on bounded smooth domain Ω of $\mathbb{R}^N$ with $\Omega = A \cup B$ where A and B are both nonempty sets. We assume a fashion rescaling equation and prove that, in the L^1-norm, the solutions ϵ-sequence of rescaling equation converges to the solution of Kirchhoff equation, under compactibility condiction on the intersection of A and B. We deal with Dirichlet boundary condiction.

1 Introduction

Our main goal is to deal with the nonlocal diffusion equation that involve three differents smooth kernels on bounded smooth domain Ω of $\mathbb{R}^N$ with $\Omega = A \cup B$ where A and B are both nonempty sets, the parabolic version of [2]. The dinamical interpretation of the following equation is that the involves smooth kernerls works as: one controls the jump from A to A, the second one controls the jumps from B to B and the third one governs the interactions between A and B. We deal with Dirichlet boundary conditions, which is given by the following formulation if $x \in \Omega$ and $t > 0$

$$\begin{aligned} u_t(x,t) = \chi_A(x)\int_A J(x-y)(u(y,t)-u(x,t))\,dy + \chi_B(x)\int_A G(x-y)(u(y,t)-u(x,t))\,dy \\ +\chi_A(x)\int_B G(x-y)(u(y,t)-u(x,t))\,dy + \chi_B(x)\int_B K(x-y)(u(y,t)-u(x,t))\,dy, \end{aligned} \tag{1}$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) = u_0(x), x \in \Omega. \\ u(x,t) = 0, x \notin \Omega \end{aligned} \tag{2}$$

Above, $\chi_.$ denote the characteristic functions of $. = A$ or $. = B$. Note that the equation can also be written as a system on $x \in A$ or $x \in B$. By the Banach Fixed Point, we guarantee that (1) has unicity of solution in $C([0,T];L^1(\Omega))$ and has comparison propriety of the solutions also valid, since the hypotesis on the kernels $\tilde{V} = J, K, G$ and the domain Ω is given by

1. On the domain: Let Ω a bounded smooth domain, which is split by two nonempty subsets A and B.
2. On the kernels of type $\tilde{V} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a nonnegative continuous radial function with $\tilde{V}(0) > 0$ and unitary integral in $\mathbb{R}^N$.

2 Main Results

The main result is to show that the following Kirchhoff parabolic equation can be effectively approximated by our fashion nonlocal problems structured (1), with adjusted kernels and a bounded domain $\Omega := A \cup B \subset \mathbb{R}$, assuming the particular case $A = (-1, 0]$ and $B = [0, 1)$.

$$\begin{cases} v_t(x,t) = a_J v_{xx}(x,t) & x \in int(A), t > 0 \\ v_t(x,t) = a_K v_{xx}(x,t) & x \in int(B), t > 0 \\ v(x,t) = 0, & x \in \partial\overline{(A \cup B)}, t > 0 \\ v(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3}$$

This involves showing that as certain parameters, such as the rescaled kernels, are modified, the solutions to these nonlocal problems converge in $L^1(\Omega)$-norm towards the solution of the Dirichlet Problem for the couple heat equation, with appropriated constant of diffusion, employing techniques cited by [1]. Assuming that the support of the function is $(-1, 1)$, the rescaling of the kernels is given by $V = J, K : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \to \mathbb{R}$

$$V_\epsilon(\xi) = \frac{1}{\epsilon} V\left(\frac{\xi}{\epsilon}\right), \quad G_\epsilon(\xi) = G\left(\frac{\xi}{\epsilon^2}\right).$$

The diffusion coefficients defined by

$$a_V = \frac{1}{2}\int_{\mathbb{R}} V(z) z^2 dz.$$

We will delve into a specific particular domain, accompanied by an integral condition, the compatibility condiction on $x = 0$:

$$\int_0^\infty K(z) z dz \frac{u_x(0^-)}{\epsilon} = \int_0^\infty J(z) z dz \frac{u_x(0^+)}{\epsilon}$$

Since we are dealing with Dirichlet condiction, we have the rescaled problem, for each $\epsilon > 0$:

$$\begin{cases} (u^\epsilon)_t(x,t) & = \frac{\chi_A(x)}{\epsilon^2}\int_{-\infty}^0 J_\epsilon(x-y)(u^\epsilon(y,t) - u^\epsilon(x,t))dy + \frac{\chi_B(x)}{\epsilon^2}\int_{-\infty}^0 G_\epsilon(x-y)(u^\epsilon(y,t) - u^\epsilon(x,t))\,dy + \\ & \frac{\chi_A(x)}{\epsilon^2}\int_0^\infty G_\epsilon(x-y)(u^\epsilon(y,t) - u^\epsilon(x,t))dy + \frac{\chi_B(x)}{\epsilon^2}\int_0^\infty K_\epsilon(x-y)(u^\epsilon(y,t) - u^\epsilon(x,t))\,dy, x \in \Omega,\ t > 0 \\ u^\epsilon(x,0) & = u_0(x), \qquad x \in \Omega. \end{cases} \tag{4}$$

Our aim is to show that the solution of the equation above converges to the following heat equation in the $L^1(\Omega)$ norm, if the solution of following equation v is of class $C^{2+\alpha, 1+\frac{\alpha}{2}}$ in $int(A)$ and $int(B)$, for $0 < \alpha < 1$. We will prove the following Theorem, considering $L^1(\Omega)$-norm:

Theorem 2.1 (Convergence of the Rescaling Kernel). *Let* $A = (-1, 0]$ *and* $B = [0, 1)$, $0 < \alpha < 1$ *and* $v \in C^{2+\alpha, 1+\frac{\alpha}{2}}((int(A) \cup int(B)) \times [0, T])$ *the solution of* (3). *Let* u_ϵ *be the solution of the rescaled problem* (4) *for each* $\epsilon > 0$. *Then*

$$\sup_{t \in [0,T]} \|u_\epsilon(.,t) - v(.,t)\|_{L^1(\Omega)} \to 0,\ \epsilon \to 0.$$

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FRACTIONAL DIFFUSION-WAVE EQUATIONS WITH CRITICAL NONLINEARITY IN LEBESGUE SPACES

BRUNO DE ANDRADE[1], CLAUDIO CUEVAS[2] & MASTERSON COSTA[2]

[1]Federal University of Sergipe - UFS, DM, SE, Brazil, bruno@mat.ufs.br

[2]Federal University of Pernambuco - UFPE, DM, Brazil, cch@dmat.ufpe.br, masterson.costa@gmail.com

Abstract

In this work, we study several issues related to the fractional diffusion-wave problem, with critical nonlinearity in the interpolation and extrapolation scales applied in Lebesgue space. We prove the local existence of ε-regular mild solutions to the problem in time, uniqueness and continuous dependence on the initial data, as well as their possible continuation for a maximum interval of existence and an alternative explosion of solutions that satisfy a specific condition of controlled behavior at $t = 0$.

1 Introduction

It is well known that the operator $-\Delta_x$ with Dirichlet boundary conditions can be seen as a sectorial operator in $E_q^0 := L^q(\Omega)$ with domain $E_q^1 := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ (see, [3]). Therefore, we can associate this operator with fractional power spaces $\{X_q^\gamma\}_{\gamma\in\mathbb{R}}$ (see, [1]). Let $\mathcal{A}_q := \Delta_x$ be the Laplacian operator defined in fractional power scales, considering the rescaling $X_q^\gamma := E_q^{\gamma-1}$, $\gamma \in \mathbb{R}$, where $\mathcal{A}_q : X^1 \subset X_q^0 \to X_q^0$ is the E^{-1} realization of the Laplacian operator.

The formulation of the diffusion-waves equations is given by

$$\begin{cases} \partial_t^\alpha u = \mathcal{A}_q u + |u|^{\rho-1}u, \text{ in } [0,\infty)\times\Omega, \\ u = 0, \text{ on } [0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), \ u'(0,x) = u_1(x), \text{ in } \Omega, \end{cases} \tag{1}$$

where $\rho = 1 + \frac{2q}{N}$ is the Sobolev critical exponent, and Ω is an open subset of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$, ∂_t^α is the Caputo fractional derivative of order $\alpha \in (1,2)$, the initial conditions $u(0) = u_0, \ u'(0) = u_1 \in L^q(\Omega)$. The technique to solve the problem comes from using Sobolev-type embeddings involving potential Bessel spaces $H_p^s(\Omega)$. (see, [2]).

$$\begin{cases} E_q^\gamma \hookrightarrow H_q^{2\gamma}(\Omega), & \gamma \geq 0, \quad 1 < q < \infty, \\ E_q^{-\gamma} = (E_{q'}^\gamma)', & \gamma \geq 0, \quad 1 < q < \infty, \ q' = \frac{q}{q-1}. \end{cases}$$

In the work carried out in Naldisson's Thesis (see, [4]), the author provides the solution to this problem (1) for the subcritical case. The result presented here is the critical version for this problem.

2 Main Results

Theorem 2.1. *Let* $\alpha \in (1, \frac{2\phi_q}{\pi})$*,* $1 < q, \rho < \infty$*, such that,* $q = \frac{N(\rho-1)}{2}$*, taking an arbitrary* $v_0 \in L^q(\Omega)$*, there are positive values small enough* r, τ_0 *such that for any initial data* $u_0, u_1 \in B_r(v_0) \subset L^q(\Omega)$*, where* $q > \frac{N}{N-2}$*,* $0 < \varepsilon < \frac{N}{N+2q}$ *and* $\gamma(\varepsilon) = \rho\varepsilon$*, there exists a unique* ε*-regular local mild solution for the problem* (1)*. Furthermore, it follows that, for* $\theta \in (0, \rho\varepsilon)$ *we have*

$$\lim_{t \to 0^+} t^{\alpha\theta} \|u(t, u_0, u_1)\|_{X^{1+\theta}} = 0.$$

Moreover, if $u_0, u_1, w_0, w_1 \in B_r(v_0) \subset L^q(\Omega)$*, the there exists a constant* $C > 0$ *such that*

$$t^{\alpha\theta} \|u(t, u_0, u_1) - u(t, w_0, w_1)\|_{X^{1+\theta}} \leq C(\|u_0 - w_0\|_{L^q(\Omega)} + \|u_1 - w_1\|_{L^q(\Omega)}),$$

for all $0 \leq \theta < \rho\varepsilon$ *and* $\forall t \in [0, \tau_0]$*. The* ε*-regular mild solution* $u(t, u_0, u_1)$ *can be continued on an interval* $[0, \tau_{max})$*, where* $\tau_{max} \in (\tau_0, \infty]$*. If* $\tau_{max} < \infty$*, then*

$$\limsup_{t \to \tau_{max}^-} \|u(t, u_0, u_1)\|_{X^{1+\varepsilon}} = \infty.$$

Moreover, if $\bar{u}$ *is a* ε*-regular mild solution on some interval* $[0, \tau_1]$ *for the problem* (1) *satisfying*

$$\lim_{t \to 0^+} t^{\alpha\varepsilon} \|\bar{u}(t)\|_{X^{1+\varepsilon}} = 0,$$

then $\tau_1 < \tau_{max}$ *and* $\bar{u}(t) = u(t, u_0, u_1)$ *for all* $t \in [0, \tau_1]$*.*

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RAPID BOUNDARY STABILIZATION OF THE LONGITUDINAL VIBRATIONS OF A BAR

M. MILLA MIRANDA[1] & A. T. LOUREDO[2]

[1]UFRJ, RJ, Brazil, mmillamiranda@gmail.com
[2]DM, UEPB, PB, Brazil, aldolouredo@gmail.com

Abstract

This paper is concerned with the exponential decay, with an arbitrary decay rate $\omega > 0$, of the longitudinal vibrations of a bar. The result is obtained following the ideas of Komornik[1]

1 Introduction

We consider the system

$$\left| \begin{array}{l} y''(x,t) - y_{xx}(x,t) = 0,\ 0 < x < L,\ t > 0, \\ y(0,t) = 0,\ y''(L,t) + y_x(L,t) = u(t),\ t > 0, \\ y(x,0) = y^0(x),\ y'(x,0) = y^1(x),\ 0 < x < L, \end{array} \right. \tag{1}$$

where $u(t)$ is a control that acts on the system at the end $x = L$ of the interval $(0, L)$. The mathematical deduction of (1) with $u = 0$ can be found in M.Milla Miranda et al.[2]

The usual inner product and norm of the space $L^2(0, L)$ are denoted by (u, v) and $|u|$. Let V be the complex Hilbert space

$$V = \{u \in H^1(0, L); u(0) = 0\}$$

equipped with the inner product

$$((u, v)) = \int_0^L Du(x)\overline{Dv(x)}dx \quad (Du(x) = \frac{du(x)}{dx})$$

and norm $\|u\| = ((u, u))^{\frac{1}{2}}$.

Introduce the Hilbert space $L^2_*(0, L) = L^2(0, L) \times \mathbb{C}$ with the usual product of spaces. It is constructed a closed subspace Z of $H^2(0, L)$ where it is possible to define $D^2u(L)$ satisfying $D^2u(L) = -Du(L)$. With identifications of Hilbert spaces it is obtained

$$Z \hookrightarrow V \hookrightarrow L^2_*(0, L) \hookrightarrow V' \hookrightarrow Z'$$

Consider the Hilbert spaces $H = V' \times L^2_*(0, L), H' = V \times L^2_*(0, L)$ and the operators

$$\begin{array}{l} A : D(A) \subset H \longrightarrow H, Az = A[\sigma, y] = [-D^2y, -\sigma] \\ A^* : D(A^*) \subset H' \longrightarrow H', A^*\varphi = A^*[\eta, \gamma] = [-\gamma, -D^2\eta] \end{array}$$

where

$$\begin{array}{l} D(A) = \{[\sigma, y]; \sigma \in V, y \in Z\} \\ D(A^*) = \{[\eta, \gamma]; \eta \in Z, \gamma \in V\} \end{array}$$

Consider the adjoint problem of (1):

$$\varphi' = -A^*\varphi, \varphi(0) = \varphi^0, \varphi^0 = [\eta^0, \gamma^0]. \tag{2}$$

Introduce the control spaces

$$G = \{[D^2\eta(L), \gamma(L)] \in \mathbb{C}^2; [\eta, \gamma] \in D(A^*)\}$$

and $G' = G$. Set the operators

$$B^* : D(A^*) \longrightarrow G', B^*[\eta, \gamma] = [D^2\eta(L), \gamma(L)]$$

and $B = B^{**}$.

We prove that the four hypotheses required in order to apply the result of Komornik [1] are satisfied. In particular A^* generates a continuous group on H'. Therefore the solution φ of (2) is given by $\varphi(t) = e^{-tA^*}\varphi^0, \varphi^0 \in D(A^*)$

Give a real number $\omega > 0$. Consider $T_\omega = T + (2\omega)^{-1}$ where $T > T_0$, $T_0 = 4(L - x^0), x^0 \leq 0$, and consider also the real function $e_\omega(t)$ given in Komornik [1]. Define the operator $\Lambda_\omega : H' \longrightarrow H$ given by

$$< \Lambda_\omega\varphi^0, \psi^0 >_{H,H'} = \int_0^{T_\omega} e_\omega(t) < JB^*e^{-tA^*}\varphi^0, B^*e^{-tA^*}\psi^0 >_{G,G'} dt, \quad \varphi^0, \psi^0 \in H'$$

where $J : G' \longrightarrow G$ denotes the canonical Riesz isomorphism. It is proved that Λ_ω in an isomorphism of H' onto H. Therefore

$$||z||_W =< \Lambda_\omega^{-1}z, z >^{1/2}_{H',H} \quad , z \in H$$

defines an equivalent norm in H.

2 Main Result

By applying Theorem 3.1 of Komornik [1], we obtain the following result:

Theorem 2.1. *Consider $T > T_0$, $T_0 = 4(L - x^0), x^0 \leq 0$ and arbitrary real number $\omega > 0$. Set $T_\omega = T + (2\omega)^{-1}$ and $F = -JB^*\Lambda_\omega^{-1}$. Then the operator $\widetilde{A} + BF$ is the generator of a continuous group in H and the solution z of the closed-loop problem*

$$z' = \widetilde{A}z + BFz, z(0) = z^0, z^0 \in H$$

satisfies the estimate

$$||z(t)||_W \leq ||z^0||_W e^{-\omega t}, \forall z^0 \in H, \forall t \geq 0$$

Remark 2.1. *The operator $\tilde{A}$ is the extension of the operator A and its domain is given by $\Lambda_\omega D(A^*)$*

As $||.||_W$ and $||.||_H$ are equivalent in H, the above inequality provides

$$||z(t)||_H \leq C||z^0||_H e^{-\omega t}, \forall z^0 \in H, \forall t \geq 0.$$

where $C > 0$ is a constant independent of z^0.

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BI-PARAMETER PATHWISE-PROBABILITY-EXPECTATION ROBUSTNESS OF RANDOM ATTRACTORS FOR NONAUTONOMOUS STOCHASTIC LAMÉ SYSTEMS ON UNBOUNDED DOMAINS

RENHAI WANG[1], MIRELSON M. FREITAS[2] & TRAN QUANG MINH[3]

[1]Guizhou Normal University, Guiyang, China, rwang-math@outlook.com
[2]University of Brasília, Brasília, DF, Brazil, mirelson.freitas@unb.br
[3]Vietnam National University, Ho Chi Minh City, Vietnam, tqminh.khtn93@gmail.com

Abstract

The Lamé system represents a classical model in many fields such as isotropic elasticity and seismology. The single-parameter stability of global and pullback attractors of deterministic Lamé systems on bounded domains was recently considered in the literature. In this paper, we study the bi-parameter stability of random attractors of stochastic non-autonomous Lamé systems on unbounded domains. The existence, uniqueness and periodicity of pullback random attractors $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ are established in the natural energy space $\mathcal{H} := (H^1(\mathbb{R}^N))^N \times (L^2(\mathbb{R}^N))^N$ for any dimension $N \in \mathbb{N}$. The upper semi-continuity of $\mathcal{A}_\varepsilon(\tau,\omega)$ in almost sure paths, probability and expectation is established as the time parameter τ tends to $\pm\infty$ and the noise intensity ε approaches zero simultaneously, and the limiting set is determined by the global attractor of the deterministic autonomous Lamé system.

1 Introduction

We study random attractors of Lamé systems on the unbounded domain $\mathbb{R}^N$ subject to time-dependent external forces as well as stochastic noise perturbations:

$$\begin{cases} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div} u + \alpha u_t + \beta u + f(x,u) = g(t,x) + \varepsilon u \circ \dfrac{dW}{dt}, \ t > \tau, \\ u(\tau,x) = u_0(x), \ u_t(\tau,x) = u_1(x), \quad x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $N \in \mathbb{N}$, $\tau \in \mathbb{R}$, α and β are positive constants, μ and λ are the Lamé's constants satisfying $\mu+\lambda > 0$ with $\mu > 0$, $u = (u_1, \dots, u_N)$ represents the displacement, g represents some time-dependent external force, W is an independent two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The stochastic term in (1) is understood in the sense of Stratonovich's integration, and the function f has the vector form $f(x,u) = (f_1(x,u_1), \dots, f_N(x,u_N))$ satisfying

$$|f_i(x,u_i)| \le c_1|u_i|^\gamma + \phi_{1i}(x), \quad i = 1, \dots, N, \tag{2a}$$
$$f(x,u)\cdot u - c_2F(x,u) \ge \phi_2(x), \quad \forall u \in \mathbb{R}^N, \tag{2b}$$
$$F(x,u) \ge c_3|u|^{\gamma+1} - \phi_3(x), \quad \forall u \in \mathbb{R}^N, \tag{2c}$$
$$|f_i'(x,u_i)| \le c_4|u_i|^{\gamma-1} + \phi_{4i}(x), \quad i = 1, \dots, N, \tag{2d}$$

where $f_i'(x,s) := \partial_s f_i(x,s)$, $F(x,u) := \sum_{i=1}^N \int_0^{u_i} f_i(x,y)dy$, $\phi_1 = (\phi_{11}, \dots, \phi_{1N}) \in (L^2(\mathbb{R}^N))^N$, $\phi_2 \in L^1(\mathbb{R}^N)$, $\phi_3 \in L^1(\mathbb{R}^N)$, $\phi_4 = (\phi_{41}, \dots, \phi_{4N}) \in (L^{2N}(\mathbb{R}^N))^N$, and the growth rate $\gamma \ge 1$ satisfying a subcritical growing condition: $\gamma \in [1,\infty)$ if $N = 1,2$ and $\gamma \in [1, \frac{N}{N-2})$ if $N \ge 3$.

2 Main Results

Next, we present the main results of this paper. To this end, we set some numbers:

$$\sigma_1 := \min\{\delta_1, \delta, \delta c_2\}, \ \sigma_2 := \min\left\{2(2\delta+1), \frac{2(2\delta+1)}{\delta_2}, \frac{c_1}{c_2}, 4\right\}, \ \varepsilon_0 := \min\left\{1, \frac{\sigma_1}{2(\gamma+1)^2(\frac{2}{\sqrt{\pi\delta}}+\frac{1}{\delta})\sigma_2}\right\},$$

where $\delta > 0$ is a constant such that $\delta_1 := \alpha - \delta > 0$ and $\delta_2 := \beta + \delta^2 - \alpha\delta > 0$.

Theorem 2.1. *(Main results I) Suppose that assumptions (2a)-(2d) and the following condition*

$$\int_{-\infty}^{\tau} e^{\frac{1}{2(\gamma+1)^2}\sigma_1 r}\|g(r)\|_2^2 dr < \infty, \quad \forall \tau \in \mathbb{R}, \tag{1}$$

hold. Then, for all $\varepsilon \in (0, \varepsilon_0]$, the cocycle Φ_ε associated with the non-autonomous stochastic Lamé system (1) has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A}_\varepsilon = \{\mathcal{A}_\varepsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ in $\mathcal{H} := (H^1(\mathbb{R}^N))^N \times (L^2(\mathbb{R}^N))^N$, in the sense of Caraballo et al. [1] and Wang [2, 3], such that

$$\mathcal{A}_\varepsilon(\tau,\omega) = \bigcap_{t_0\geq 0}\overline{\bigcup_{t\geq t_0}\Phi_\varepsilon(t, \tau - t, \theta_{-t}\omega, K_\varepsilon(\tau - t, \theta_{-t}\omega))}, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega, \tag{2}$$

where $\mathcal{D}$ is a collection of some nonempty random subsets of $\mathcal{H}$ with a tempered growing rate no more than the growing rate of the exponential function $e^{\frac{1}{(\gamma+1)^2}\sigma_1 t}$ as $t \to +\infty$. If, in addition, the time-dependent force g is T-periodic with period $T > 0$, then the attractor $\mathcal{A}_\varepsilon$ is also T-periodic with period T, i.e., $\mathcal{A}_\varepsilon(\tau + T, \omega) = \mathcal{A}_\varepsilon(\tau,\omega)$ for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

Theorem 2.2. *(Main results II) Suppose that assumptions (2a)-(2d) and (1) hold. If the time-dependent external force $g \in L^2_{\text{loc}}(\mathbb{R}; (L^2(\mathbb{R}^N))^N)$ converges to a time-independent function g_∞ in the sense that*

$$\lim_{\tau\to+\infty}\int_\tau^\infty \|g(r) - g_\infty\|_2^2 dr = 0, \tag{3}$$

then, the section $\mathcal{A}_\varepsilon(\tau,\omega)$ is upper semicontinuous to the global attractor $\mathcal{A}_\infty$ of the autonomous deterministic Lamé system (i.e., (1) with $\varepsilon = 0$ and $g(t) \equiv g_\infty$) as $\tau \to -\infty$ and $\varepsilon \to 0$ simultaneously:

(i) Convergence in almost paths*:*

$$\lim_{\tau\to+\infty}\lim_{\varepsilon\to 0}\text{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon(\tau,\omega), \mathcal{A}_\infty) = 0, \ \mathbb{P}\text{-}a.s.. \tag{4}$$

(ii) Convergence in probability*:*

$$\lim_{\tau\to+\infty}\lim_{\varepsilon\to 0}\mathbb{P}\{\omega \in \Omega : \text{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon(\tau,\omega), \mathcal{A}_\infty) \geq \varpi\} = 0, \quad \forall\, \varpi > 0. \tag{5}$$

(iii) Convergence in expectation*:*

$$\lim_{\tau\to+\infty}\lim_{\varepsilon\to 0}\int_\Omega \text{dist}_{\mathcal{H}}(\mathcal{A}_\varepsilon(\tau,\omega), \mathcal{A}_\infty) d\mathbb{P}(\omega) = 0. \tag{6}$$

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FINE STRUCTURE PROPERTIES OF ENTROPY SOLUTIONS FOR SCALAR CONSERVATION LAWS

N. DANIELSKI[1] & W. NEVES[1]

[1]Universidade Federal do Rio de Janeiro, IM, RJ, Brazil
nicolasdanielski@im.ufrj.br, wladimir@im.ufrj.br

Abstract

This work investigates the fine structure of entropy solutions to scalar conservation laws, building upon the works of De Lellis, Otto, Westdickenberg, and Silvestre. We consider both autonomous and non-autonomous conservation laws and explore the conditions under which the solutions exhibit BV-like properties. In particular, we examine the influence of the flux function's regularity and non-degeneracy conditions on the solutions' structure. Our study is motivated by the phenomenon observed by Oleinik in one-dimensional cases and seeks to extend these findings to higher dimensions and non-autonomous flux functions. We address the minimum regularity required for the flux function with respect to the non-autonomous variable to ensure a structure similar to that found in bounded variation functions, as well as the potential to achieve similar structural results for continuous, weakly non-degenerate flux functions in the autonomous case. .

The starting point for our study is the articles by De Lellis, Otto, Westdickenberg [1], and Silvestre [3]. In these articles, they consider the following autonomous scalar conservation law

$$\operatorname{div}_x f(u(x)) = 0 \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^{n+1}$ is an open set. When $\Omega = (0, +\infty) \times \mathbb{R}^n$ and the strong trace u^τ of an entropy solution $u \in L^\infty(\Omega)$ of (1) has locally bounded variation, then the same occurs with u due to the L^1 contraction property. Hence, the entropy solution u inherits the fine structure of functions of bounded variation. In particular, there exists a set $J \subset \Omega$ with the following properties:

- J is $\mathcal{H}^n$-rectifiable.
- Every point in $\Omega \setminus J$ is a Lebesgue point of u.
- u has, essentially, traces to the right and to the left of J.

However, when u^τ is only L^∞, the structure described above is generally lost, as can be seen when the flux function f is linear. Therefore, one may inquire how to transfer the above structure to u when u^τ is only in L^∞? The first step in answering this question was taken by Oleinik in [4], who proved the following: When $n = 1$ and f is strictly convex, every entropy solution u of (1) satisfies, in the sense of distributions,

$$u_x \leq \frac{1}{(\min f'')t} \quad \text{in } (0, \infty) \times \mathbb{R}.$$

This inequality gives the solution u a structure of bounded variation even if u^τ is only in L^∞. Unfortunately, this type of phenomenon observed by Oleinik is exclusively one-dimensional, as noted by David Hoff in [2]. However, several previous works indicated that the non-linearity of the flux function could induce some type of regularity on entropy solutions. Thus, assuming that the flux satisfies the following non-degeneration condition, that is, for each $\xi \in S^n$,

$$\mathcal{L}^1\left(\{u \in \mathbb{R};\, \xi \cdot f'(u) = 0\}\right) = 0,$$

the authors in [1] proved that entropy solutions in L^∞ possess a fine structure very close to that presented by functions of bounded variation. More precisely, they proved the following: There exists a set $J \subset \Omega$ with the following properties:

- J is $\mathcal{H}^n$-rectifiable.
- Any $y \in \Omega/J$ satisfies
$$\lim_{r\to 0} \frac{1}{r^{n+1}} \int_{B_r(y)} |u(x) - \bar{u}_{y,r}|\, dx = 0,$$
where $\bar{u}_{y,r}$ is the average of u over the ball $B_r(y)$.
- u has, essentially, traces to the right and to the left of J.

Here, in this part of the project, our study interest is the following non-autonomous scalar conservation law

$$\mathrm{div}_x f(x, u(x)) = 0 \quad \text{in } \Omega.$$

In the above context, the problems we will address are the following:

1) What minimum regularity with respect to the non-autonomous variable x must the flux function f possess so that an entropy solution in L^∞ has a structure similar to that obtained in [1]?

2) Even for the autonomous case (1), would it be possible to obtain the same structure for an entropy solution in L^∞ if the flux function is only continuous and weakly non-degenerate in the sense that, for any $\xi \in S^n$, the function $\lambda \mapsto f(\lambda) \cdot \xi$ is not constant in any non-degenerate interval?

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AN ESTIMATE FOR $\limsup_{T\to\infty} \|U(\cdot,T)\|_{L^\infty}$

P. L. GUIDOLIN[1] & J. Q. CHAGAS[2]

[1]Universidade Federal do Rio Grande do Sul , IME, RS, Brazil, patiguidolin@hotmail.com

[2]Universidade Estadual de Ponta Grossa, DEMAT, PR, Brazil, jocemarchagas@uepg.br

Abstract

In this paper a rigorous study concerning estimates for the sup norm of weak bounded solutions for one-dimensional advection-diffusion equations $u_t + \mathrm{div} f(x,t,u) = \mu(t)\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, with $0 \leq k < 2p-3$ and initial data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is provided using a technique based on energy methods.

1 Introduction

In this work, we obtain an estimate for the initial value problem for evolution p-Laplacian equations of the type

$$\begin{aligned} u_t(x,t) + \mathrm{div} f(x,t,u) &= \mu(t)\mathrm{div}(|\nabla u|^{p-2}\nabla u), \\ u(x,0) &= u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}). \end{aligned} \tag{1}$$

Here, $p > 2$ is constant, $\mu \in C^0([0,\infty))$ is everywhere positive, and $f = (f_1, f_2, ..., f_n)$ a given continuous field satisfying the growth condition

$$|f(x,t,u)| \leq F(t)|u|^{k+1} \quad \forall x \in \mathbb{R}, t \geq 0 \text{ and } u \in \mathbb{R},$$

for some $F \in C^0([0,\infty))$ and some constant $k \geq 0$.

By a (bounded) solution of (1) in some time interval $[0,T_*)$, we mean any function $u(\cdot,t) \in C^0([0,T_*), L^1_{loc}(\mathbb{R}) \cap L^p_{loc}((0,T_*), W^{1,p}_{loc}(\mathbb{R}))$ satisfying the equation (1) in $\mathcal{D}'(\mathbb{R} \times (0,T_*))$, with $u(\cdot,0) = u_0$ and $u(.,t) \in L^\infty_{loc}([0,T_*), L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, that is, for every $0 < T < T_*$ given, we have

$$\|u(\cdot,t)\|_{L^1(\mathbb{R})} \leq M_1(T), \quad \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq M_\infty(T), \quad \forall 0 \leq t \leq T$$

for some bounds $M_1(T)$, $M_\infty(T)$ depending on T (and the solution u considered). For the local (in time) existence of such solutions, see e.g. [3, 4]. In a recent work [2] the global existence was discovered when $0 \leq k < p - 2 + \frac{p-1}{n}$, where n is the dimension of the spatial variable. In addition, the following estimate was obtained

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \mathbb{U}_\infty(0,t) \leq M(n,k,p,q)\max\{\|u_0\|_{L^\infty(\mathbb{R}^n)}; \mathbb{F}_\mu(0,t)^{\frac{n}{p-1}\frac{1}{q-\sigma}}\mathbb{U}_q(0,t)^{\frac{q}{q-\sigma}}, \} \tag{2}$$

for $1 \leq q \leq \infty$, $0 < t < T_*$, some constant $M(n,k,p,q)$ depending only on n,k,p,q, and where $\sigma, \mathbb{F}_\mu(0,t), \mathbb{U}_q(0,t), \mathbb{U}_\infty(0,t)$ are defined below

$$\sigma = \frac{n(k-p+2)}{p-1}, \quad \mathbb{F}_\mu(0,t) = \sup_{0<\tau<t} \frac{F(\tau)}{\mu(\tau)}, \quad \mathbb{U}_q(0,t) = \sup_{0<\tau<t} \|u(\cdot,\tau)\|_{L^q(\mathbb{R}^n)}. \tag{3}$$

The objective of this work is to understand the behavior of the function when $t \to \infty$, analyze under what conditions we guarantee that the solution remains bounded when $t \to \infty$. This way, using properties obtained from [2] combined with an inequality valid only in one dimension

$$\|u\|_{L^\infty(\mathbb{R})} \leq C_\infty \|u\|_{L^1(\mathbb{R})}^{1/3} \|u_x\|_{L^2(\mathbb{R})}^{2/3},$$

where $C_\infty = (3/4)^{2/3}$ (to see [1]), it is possible to find an estimate for $\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$.

2 Main Results

In one dimension, the estimate (2) can be improved to derive an estimate for the limit $\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$. From this point on, it will be convenient to introduce $\mathcal{F}_\mu$, $\mathcal{U}_q$ given by

$$\mathcal{F}_\mu = \limsup_{t\to\infty} \frac{F(t)}{\mu(t)} \text{ and } \mathcal{U}_q = \limsup_{t\to\infty} \|u(\cdot,t)\|_{L^q(\mathbb{R})}.$$

Theorem 2.1. *Let $u(\cdot,t)$ be a solution of the problem (1), $q \geq 1$, $0 \leq k < q(p-1)+p-2$, and $\mathcal{F}_\mu$, $\mathcal{U}_q$ defined as above, then*

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq M(C_\infty,k,p,q)\mathcal{F}_\mu^{\frac{1}{p-1}\frac{1}{q-\sigma}} \mathcal{U}_q^{\frac{q}{q-\sigma}}, \tag{1}$$

where σ is defined in (3).

In particular, in the important case $q = 1$ considered above, from decay of the L^1 norm (to see [2]), the estimate (1) stays

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq M(C_\infty,k,p)\mathcal{F}_\mu^{\frac{1}{p-1}\frac{1}{1-\sigma}} \|u_0\|_{L^1(\mathbb{R})}^{\frac{1}{1-\sigma}}.$$

In this case, $u(\cdot,t)$ has a uniform limitation for all time $t > 0$.

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SUSPENSION BRIDGE WITH INTERNAL DAMPING OF FRACTIONAL DERIVATIVE TYPE

RAFAEL O. DE JESUS[1,3], CARLOS A. RAPOSO[2], JOILSON O. RIBEIRO[3] & ISAAC ELISHAKOFF[4]

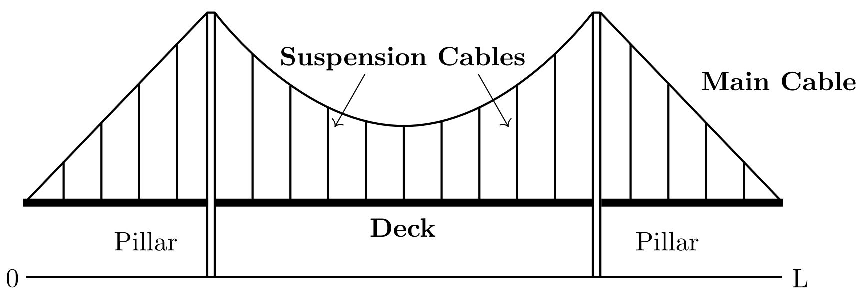
[1]Universidade de Pernambuco, CM, Petrolina, PE, Brazil, rafael.oliveira@upe.br
[2]Universidade Federal do Pará, FM, Salinópolis, PA, Brazil, carlos.raposo@ufba.br
[3]Universidade Federal da Bahia, IME, Salvador, BA, Brazil, joilsonor@ufba.br
[4]Florida Atlantic University, DOME, Boca Raton, AFI, USA, elishako@fau.edu

Abstract

This manuscript deals with the well-posedness and polynomial stability of a suspension bridge system with a deck modeled by Timoshenko-Ehrenfest beam theory and under action to internal dissipations of the fractional derivative type. The existence and uniqueness of the solution are obtained by applying the Lumer-Phillips Theorem and proving the polynomial stability using the Borichev-Tomilov theorem.

1 Introduction

A suspension bridge is a mechanical structure that carries vertical loads through the main cables modeled by an elastic string $u = u(x, t)$, coupled to the deck employing suspension cables.



The system for a suspension bridge, where Timoshenko's theory models the deck, is given by

$$u_{tt} - au_{xx} - \tau(\phi - u) = 0,$$
$$\rho_1\phi_{tt} - k(\phi_x + \psi)_x + \tau(\phi - u) = 0,$$
$$\rho_2\psi_{tt} - b\psi_{xx} + k(\phi_x + \psi) = 0.$$

Many researchers use Thimoshenko's theory to understand how these structures behave. Arioli and Gazzola [1], in 2015, suggested a new model for the dynamics of a suspension bridge through a system of nonlinear, nonlocal hyperbolic differential equations where the equations are of second and fourth order and describe the behavior of the main components of the bridge: the deck, the sustaining cables, and the connecting hangers. In 2020, Bochicchio et al. [2] studied a linear problem of the vibrations of a coupled suspension bridge as a thermoelastic beam given by Fourier law, where the deck is modeled by the Timoshenko-Ehrenfest theory. In 2023, the existence and uniqueness of a suspension bridge modeled by Timoshenko-Ehrenfest theory were proved, with internal damping, obtaining, beyond the exponential decay, the analyticity of the solution. See [3].

In this work we consider the following model of suspension bridge with internal damping of fractional order

$$u_{tt} - au_{xx} - \tau(\phi - u) + c_1\partial_t^{\alpha,\eta} u = 0, \tag{1}$$

$$\rho_1\phi_{tt} - k(\phi_x + \psi)_x + \tau(\phi - u) + c_2\partial_t^{\beta,\zeta}\phi = 0, \tag{2}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + k(\phi_x + \psi) + c_3\partial_t^{\theta,\xi}\psi = 0. \tag{3}$$

System (1)-(3) is subject Dirichlet boundary conditions and to initial data:

$$\begin{cases} u(0,t) = u(L,t) = 0,\ t \geq 0, \\ \phi(0,t) = \phi(L,t) = 0,\ t \geq 0, \\ \psi(0,t) = \psi(L,t) = 0,\ t \geq 0. \end{cases} \tag{4}$$

$$\begin{cases} u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x),\ x \in (0,L), \\ \phi(x,0) = \phi_0(x), \ \ \phi_t(x,0) = \phi_1(x),\ x \in (0,L), \\ \psi(x,0) = \psi_0(x), \ \ \psi_t(x,0) = \psi_1(x),\ x \in (0,L), \end{cases} \tag{5}$$

The dampers used are of the type fractional integro-differential operators with exponential weight, i.e.

$$\partial_t^{\omega,\xi} f(t) = \frac{1}{\Gamma(1-\omega)}\int_0^t (t-s)^{-\omega} e^{-\xi(t-s)} f'(s)\, ds,\ (0 < \omega < 1, \xi \geq 0 \text{ and } f \in W^1([0,L))$$

For to prove the well-posed we write the equations as augmented system and transform problem (1) − (5) into the following abstract Cauchy problem

$$U_t = \begin{pmatrix} u_t \\ v_t \\ \phi_t \\ w_t \\ \psi_t \\ z_t \\ (\varphi_1)_t \\ (\varphi_2)_t \\ (\varphi_3)_t \end{pmatrix} = \begin{pmatrix} v \\ au_{xx} + \tau(\phi - u) - \gamma_1 \int_{\mathbb{R}} p(y)\varphi_1(y)dy \\ w \\ \frac{1}{\rho_1}\left[k(\phi_x + \psi)_x - \tau(\phi - u) - \gamma_2 \int_{\mathbb{R}} q(y)\varphi_2(y)dy\right] \\ z \\ \frac{1}{\rho_2}\left[b\psi_{xx} - k(\phi_x + \psi) - \gamma_3 \int_{\mathbb{R}} r(y)\varphi_3(y)dy\right] \\ -(|y|^2 + \eta)\varphi_1(y) + p(y)v \\ -(|y|^2 + \zeta)\varphi_2(y) + q(y)w \\ -(|y|^2 + \xi)\varphi_3(y) + r(y)z \end{pmatrix} = \mathcal{A}U, \quad U(0) = U_0 = \begin{pmatrix} u_0 \\ u_1 \\ \phi_0 \\ \phi_1 \\ \psi_0 \\ \psi_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6}$$

2 Main Results

Theorem 2.1. *If $U_0 \in \mathcal{H} := [H_0^1(0,L) \times L^2(0,L)]^3 \times [L^2(\mathbb{R};\, L^2(0,\, L))]^3$, then the Cauchy problem (6) exists and admits a unique weak solution $U \in C^0([0,+\infty);\mathcal{H})$ given by $U(t) = e^{t\mathcal{A}}U_0$. If $U_0 \in \mathcal{D}(\mathcal{A})$, then the obtained solution is a strong solution with the following regularity*

$$U \in C^0([0,+\infty);\mathcal{D}(\mathcal{A})) \cap C^1([0,+\infty);\mathcal{H}).$$

Theorem 2.2. *For $U_0 \in \mathcal{D}(\mathcal{A})$ and $\eta, \zeta, \xi > 0$, o C_0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ defined by the Cauchy problem (6) is polynomially stable, that is,*

$$||e^{t\mathcal{A}}U_0|| \leq \frac{C}{t^\omega}\|U_0\|_{\mathcal{D}(\mathcal{A})},\ t > 0$$

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BOUNDARY NULL CONTROLLABILITY OF DEGENERATE WAVE EQUATION AS THE LIMIT OF INTERNAL CONTROLLABILITY

BRUNO S. V. ARAÚJO[1], REGINALDO DEMARQUE[2] & LUIZ VIANA[3]

[1]Unidade Acadêmica de Matemática, Universidade Federal de Campina Grande, Campina Grande, PB, Brazil, bsergio@mat.ufcg.edu.br
[2]Departamento de Ciências da Natureza, Universidade Federal Fluminense, Rio das Ostras, RJ, Brazil, reginaldo@id.uff.br
[3]Departamento de Análise, Universidade Federal Fluminense, Niterói, RJ, Brazil, luizviana@id.uff.br

Abstract

This work is concerned with the possibility of proving the boundary null controllability for the degenerate wave equation, developing the asymptotic analysis of a suitable family of state-control pairs $((u_\varepsilon, v_\varepsilon))_{\varepsilon>0}$, solving related internal null controllability problems. The passage to the limit argument will be rigorously performed through the obtainment of a refined observabilitity type inequality, with a constant explicitly given in terms of $\varepsilon > 0$. This represents an essential point, since will allow us to achieve our required weak convergence results.

1 Introduction

Given $T > 0$, let $Q = (0,T) \times (0,1)$. For each $\varepsilon \in (0,1)$, let us consider $\omega_\varepsilon := (1-\varepsilon, 1) \subset (0,1)$. In this work, for a fixed initial data (u^0, u^1), we intend to obtain a family of distributed state-controls pairs $((u_\varepsilon, v_\varepsilon))_{\varepsilon>0}$ solving

$$\begin{cases} u_{\varepsilon tt} - (x^\alpha u_{\varepsilon x})_x = v_\varepsilon \chi_{\omega_\varepsilon}, & (t,x) \in Q, \\ \mathcal{B}u(t,0) = u_\varepsilon(t,1) = 0, & t \in (0,T), \\ u_\varepsilon(0,x) = u^0(x), u_{\varepsilon t}(0,x) = u^1(x), & x \in (0,1), \\ u_\varepsilon(T,x) = u_{\varepsilon t}(T,x) = 0, & x \in (0,1), \end{cases} \tag{1}$$

where $\mathcal{B}u(t,0) = 0$ is the suitable boundary condition related to the degenerate operator at $x = 0$, defined by

$$\mathcal{B}u(t,0) = \begin{cases} u(t,0) = 0, & \text{if } \alpha \in (0,1), \\ \lim\limits_{x\to 0+} (x^\alpha u_x)(t,x) = 0, & \text{if } \alpha \in [1,2), \end{cases}, t \in (0,T).$$

We prove that $((u_\varepsilon, v_\varepsilon))_{\varepsilon>0}$ converges to (u,h) in a suitable function space, as $\varepsilon \to 0$, with (u,h) satisfying

$$\begin{cases} u_{tt} - (x^\alpha u_x)_x = 0, & (t,x) \in Q, \\ \mathcal{B}u(t,0) = 0, \ u(t,1) = h(t), & \text{in } (0,T), \\ u(0,x) = u^0(x), u_t(0,x) = u^1(x), & x \in (0,1), \\ u(T,x) = u_t(T,x) = 0, \ \forall x \in (0,1). \end{cases} \tag{2}$$

Roughly speaking, the family $((u_\varepsilon, v_\varepsilon))_{\varepsilon>0}$ will be obtained using the well-known *Hilbert uniqueness method* (HUM), where v_ε is a solution, in the sense of transposition given by Lions-Magenes (see [2, page 47]), of the homogeneous adjoint problem associated to (1). By using the rescaling $v_\varepsilon = \frac{1}{\varepsilon^3}\varphi_\varepsilon$, we will be able to prove that $\varphi_\varepsilon \overset{*}{\rightharpoonup} \varphi$ in $L^\infty(0,T;L^2(0,1)$. Consequently, we will achieve the desired convergence of $((u_\varepsilon, v_\varepsilon))_{\varepsilon>0}$ to (u,h), where $v_\varepsilon = \frac{1}{\varepsilon^3}\varphi_\varepsilon$ and $h = -\frac{1}{3}\varphi_x(t,1)$.

2 Main Results

In the following, we denote by H^1_α the weighted Sobolev space that are naturally associated with the degenerate operator $Au = (x^\alpha u_x)_x$, the precise definition can be seen in [1]. For any $\alpha \in (0,2)$, let us set $T_\alpha = \frac{4}{2-\alpha}$. Our first result is the following observability inequality, with the constant explicitly given in terms of ε.

Theorem 2.1. *There exists $\varepsilon_0 > 0$ with the following property: for any $T > T_\alpha$ there exists a constant $C = C(T,\alpha) > 0$ such that*

$$\|v^0\|^2_{L^2(0,1)} + \|v^1\|^2_{H^{-1}_\alpha} \leq \frac{C}{\varepsilon^3}\int_0^T\int_{1-\varepsilon}^1 |v|^2\,dxdt, \quad \forall \varepsilon \in (0,\varepsilon_0), \tag{1}$$

for any solution v of

$$\begin{cases} v_{tt} - (x^\alpha v_x)_x = 0, & (t,x) \in Q, \\ \mathcal{B}v(t,0) = v(t,1) = 0, & \text{in } (0,T), \\ v(0,\cdot) = v^0 \in L^2(0,1),\ v_t(0,\cdot) = v^1 \in H^{-1}_\alpha. \end{cases} \tag{2}$$

As a consequence we can prove the exact internal controllability of the degenerate wave equation with the control domain being $\omega = (1-\varepsilon, 1)$.

Theorem 2.2. *Given $T > T_\alpha$ and $\varepsilon \in (0,1)$, for any $(u^0,u^1) \in H^1_\alpha \times L^2(0,1)$, there exists $v_\varepsilon \in L^\infty(0,T;L^2(0,1))$, solution of (3), with initial data $(v^0_\varepsilon, v^1_\varepsilon) \in L^2(0,1) \times H^{-1}_\alpha$, and the corresponding weak solution u_ε of (1). Moreover, the identity*

$$-(v^0_\varepsilon, u^1) + \langle v^1_\varepsilon, u^0\rangle = \int_0^T\int_{1-\varepsilon}^1 v^2_\varepsilon(t,x)\,dxdt \tag{3}$$

holds and there exists a constant $C = C(T,\alpha) > 0$ such that

$$\|v^0_\varepsilon\|_{L^2(0,1)} + \|v^1_\varepsilon\|_{H^{-1}_\alpha} \leq \frac{C}{\varepsilon^3}\left(\|u^0\|^2_{H^1_\alpha} + \|u^1\|^2_{L^2(0,1)}\right)^{1/2} \text{ and } \int_0^T\int_{1-\varepsilon}^1 v^2_\varepsilon\,dxdt \leq \frac{C}{\varepsilon^3}\left(\|u^0\|^2_{H^1_\alpha} + \|u^1\|^2_{L^2(0,1)}\right). \tag{4}$$

This kind of result was originally proved by Zuazua in [2, Chapitre VII, section 2.3], for the n-dimensional wave equation with the control domain being a neighborhood of the boundary. For degenerate wave equation, it was proved in [3], when $\alpha = (0,1)$. Our result holds for $\alpha \in (0,2)$, but under the restriction $\omega_\varepsilon = (1-\varepsilon, 1)$, the question remains open for a general control domain $\omega \subset\subset (0,1)$.

Finally, inequalities (4), will allow us to obtain the convergence of the family $((u_\varepsilon, v_\varepsilon))_{\varepsilon>0}$.

Theorem 2.3. *Given $T > T_\alpha$ and $\varepsilon > 0$, for any $(u^0,u^1) \in H^1_\alpha(0,1) \times L^2(0,1)$, there exist $\varphi_\varepsilon \in L^2((0,T)\times\omega_\varepsilon)$ and $u_\varepsilon \in C([0,T];H^1_\alpha) \cap C^1([0,T];L^2(0,1))$, such that:*

(a) u_ε is a weak solution of (1), with $v_\varepsilon := \frac{1}{\varepsilon^3}\varphi_\varepsilon$;

(b) $u_\varepsilon \rightharpoonup u$ and $\varphi_\varepsilon \overset{*}{\rightharpoonup} \varphi$ in $L^\infty(0,T;L^2(0,1))$, as $\varepsilon \to 0$. Moreover, u solves (2), in the sense of transposition, with $h(t) = -\frac{1}{3}\varphi_x(t,1) \in L^2(0,T)$.

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SUSPENSION BRIDGE IN VON KÁRMÁN THEORY

ROSEANE MARTINS[1], CARLOS RAPOSO[2], JOILSON RIBEIRO[1] & MIRELSON FREITAS[3]

[1] Federal University of Bahia, IME, BA, Brazil, roseane.smartins@hotmail.com, joilsonor@ufba.br
[2]Federal University of Pará, PA, Brazil, carlos.raposo@ufba.br
[3]Federal University of Brasília, MAT, DF, Brazil, mirelson@mat.unb.br

Abstract

This manuscript introduces a suspension bridge where the deck is modeled by von Kármán theory. The action of frictional damping is considered. Well-posedness is proved using the nonlinear semigroup theory and exponential stability is obtained using the energy method.

1 Introduction

In 1988, J. E. Lagnese and J. L. Lions, see [4, 5], proposed the von Kármán beam system of the type

$$\begin{cases} \rho A\omega_{tt} - EA\left[\left(u_x + \frac{1}{2}\omega_x^2\right)\omega_x\right]_x + EI\omega_{xxxx} = 0 \quad \text{in} \quad (0,\, L)\times(0,\, T), \\ \rho A u_{tt} - EA\left[u_x + \frac{1}{2}\omega_x^2\right]_x = 0 \quad \text{in} \quad (0,\, L)\times(0,\, T). \end{cases} \tag{1}$$

where $\omega(x,\, t)$ is the transverse displacement, $u(x,\, t)$ the longitudinal displacement, $(0,\, L)$ is the segment occupied by the beam, and T is a given positive time. The physical parameters represent the properties of the material being E the Young's modulus, A the cross-sectional area of the beam, L the beam length, ρA the weight per unit length and EI the beam stiffness or flexural rigidity.

In this manuscript, we consider the main cable modeled by an elastic string $v = v(x,t)$,

$$v_{tt} - \alpha v_{xx} = 0, \tag{2}$$

where the constant $\alpha > 0$ is the elastic modulus of the string (holding the main cable to the deck). The suspender cables are assumed to be linear elastic springs with standard stiffness $\lambda > 0$, which is coupled to the deck employing suspension cables, where x denotes the distance along the center line of the beam in its equilibrium configuration and t the time variable.

The coupling of (1) and (2) leads to a suspension bridge model in von Kármán theory with internal dampings given by

$$\begin{cases} v_{tt} - \alpha v_{xx} - \lambda(\omega - v) + \mu_1 v_t = 0, \text{ in } (0,\, L)\times(0,\, T) \\ \omega_{tt} - b_1\left[\left(u_x + \frac{1}{2}\omega_x^2\right)\omega_x\right]_x + b_2\omega_{xxxx} + \lambda(\omega - v) + \mu_2\omega_t = 0 \text{ in } (0,\, L)\times(0,\, T), \\ u_{tt} - b_1\left[u_x + \frac{1}{2}\omega_x^2\right]_x + \mu_3 u_t = 0 \text{ in } (0,\, L)\times(0,\, T), \end{cases} \tag{3}$$

where $\alpha, \lambda, b_1, b_2, \mu_1, \mu_2, \mu_3$ are positive and real parameters. We consider the initial data and boundary conditions, respectively

$$\begin{cases} \omega(x,\, 0) = \omega_0(x),\ \omega_t(x,\, 0) = \omega_1(x), \\ u(x,\, 0) = u_0(x),\ u_t(x,\, 0) = u_1(x), \\ v(x,\, 0) = v_0(x),\ v_t(x,\, 0) = v_1(x), \end{cases} \tag{4}$$

$$\begin{cases} u(0,\, t) = u(L,\, t) = 0, \\ \omega(0,\, t) = \omega(L,\, t) = 0, \\ \omega_x(0,\, t) = \omega_x(L,\, t) = 0, \\ v_x(0,\, t) = v_x(L,\, t) = 0. \end{cases} \tag{5}$$

In the present study, we deal with the well-posedness and asymptotic behavior of thesuspension bridge model in von Kármán theory with internal dampings given by is to prove the existence, uniqueness and exponential stability of the solution for (3)-(5). We adapt the idea as in [3], the model is posed as a nonlinear Cauchy problem

$$BU_t = \mathcal{A}U + \mathcal{F}(U) \quad \text{and} \quad U(0) = U_0, \quad \forall\, t > 0, \tag{6}$$

and shown that $B^{-1}\mathcal{A}$ generates a C_0-semigroup of contractions and $B^{-1}\mathcal{F}$ is locally Lipschitz, so follows semigroup theory for non-linear operators, (see Pazy [1], theorem 6.1.4) that there exists a unique mild solution given by

$$U(t) = e^{\mathcal{A}t}U_0 + \int_0^t e^{\mathcal{A}(t-s)}\mathcal{F}(U(s))ds,$$

and then the well-posedness is provided. We will show that the existence of weak solutions can be obtained through a regularization process and then going to the limit, by energy method we prove the exponential stability. Furthermore, for initial data taken from the generator domain, nonlinear semigroup theory also implies that the corresponding solutions are continuous in time with the values in $\overline{\mathcal{D}(B^{-1}\mathcal{A})}$ (see [2]). Thus, strong solutions satisfy $U \in C([0,\,T);\mathcal{H})$.

Our solution existence result is given by

Theorem 1.1. *If $U_0 \in \mathcal{H}$, then problem (6) has a unique mild solution $U \in C([0,\,\infty) : \mathcal{H})$ with $U(0) = U_0$. Moreover, if $U_0 \in \mathcal{D}(B^{-1}\mathcal{A})$ the mild solution is a strong solution globally defined.*

2 Main Results

Theorem 2.1. *Let $(v,\,\omega,\,u)$ be a solution of (3) where the initial data are given in $\mathcal{D}(\mathcal{A})$. Then, the energy $\mathcal{E}(t)$ satisfies*

$$\mathcal{E}(t) \le C\mathcal{E}(0)e^{-\beta t}, \quad \beta, C > 0, \quad \text{for all } t > 0.$$

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FAST ENERGY DECAY FOR DAMPED WAVE EQUATION WITHY POTENTIAL AND ROTATIONAL INERTIA TERMS

RYO IKEHATA[1] & RUY COIMBRA CHARÃO[2]

[1]Hiroshima University, HU, Japan, ikehatar@hiroshima-u.ac.jp

[2]Federal University Federal of Santa Catarina, UFSC, SC, Brazil, ruycharao@gmail.com

Abstract

We study the Cauchy problem for this model in $\mathbb{R}$ and we obtain fast energy decay and L^2-decay of the solution when $t \to \infty$. Due to we are considering this problem in one dimensional space, we can not use tools such as the Hardy and/or Poincare inequalities. This fact causes significants difficults to derive the decay property of the solutions and the energy. The potential term play a role for compensating there weak points.

1 Introduction

The Cauchy problem is given by

$$\begin{aligned} &u_{tt} - u_{ttxx} - u_{xx} + V(x)u + u_t = 0, \quad (t,\ x) \in (0,\ \infty) \times \mathbb{R} \\ &u(0,\ x) = u_0(x), \quad u_t(0,\ x) = u_1(x), \qquad x \in \mathbb{R} \end{aligned} \tag{1}$$

where $V(x) > 0$, $V \in \mathrm{BC}^1(\mathbb{R})$ and $(u_0,\ u_1) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$, and the assumption that

$$\frac{u_1 - (u_1)_{xx} + u_0}{\sqrt{V(x)}} \in L^2(\mathbb{R}).$$

An example for $V(x)$ is $V(x) = \left(1 + x^2\right)^{-\frac{\alpha}{2}}$ with $\alpha > 0$.

The condition that $f \in \mathrm{BC}(\mathbb{R})$ implies that $f(x)$ continuous and bounded in $\mathbb{R}$. $f \in \mathrm{BC}^1(\mathbb{R})$ implies f and f' in $\mathrm{BC}(\mathbb{R})$. $\|f\|$ means the usual L^2-norm of the function $f \in L^2(\mathbb{R})$.

2 Main Results

The main results are

Proposition 2.1. *Assume $V \in \mathrm{BC}^1(\mathbb{R})$ satisfies $V(x) > 0$ for $x \in \mathbb{R}$ and let $(u_0,\ u_1) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$ satisfiying*

$$\frac{u_1 - (u_1)_{xx} + u_0}{\sqrt{V(x)}} \in L^2(\mathbb{R})$$

Then, the strong solution $C^2\left([0,\ \infty)\,;\, H^2(\mathbb{R})\right)$ satisfies

$$E_u(t) = O\left(t^{-1}\right), \qquad t \to +\infty.$$

Theorem 2.1. *With the same assumption in the above Proposition 2.1, and the hyphothesis that $|V'| \le C\,V$, for $x \in \mathbb{R}$, with $C > 0$ constant, the following results holds*

$$E_u(t) \le O\left(t^{-2}\right), \quad t \to +\infty$$

and

$$\|u(t)\| \le \frac{C}{1+t}, \quad t > 0, \quad C > 0 \quad \textbf{\textit{constant,}}$$

where $E_u(t)$ is the natural energy of the system (1).

Such problem in the work has a relation with other problems for the Boussinesq linear equations for hydrodynamic models.

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SEMICLASSICAL HOMOGENIZATION OF THE WAVE EQUATION WITH OSCILLATING COEFFICIENT AND LOCALIZED INITIAL PERTURBATION

S. SERGEEV

Pontifícia Universidade Católica do Rio de Janeiro, Brazil, sergeevse1@gmail.com

Abstract

We discuss the homogenization procedure of the Cauchy problem for the wave equation with fast-oscillating coefficient and localized initial perturbation in the whole space.

The problem contains two small parameters. The first parameter $\mu \ll 1$ is the localization parameter and it describes the wavelength. The second parameter $\varepsilon \ll 1$ is the parameter of the fast oscillations of the coefficient.

We provide the adiabatic approximation procedure to obtain the homogenized equation with variable and smooth coefficients. Depending on the ratio between those two parameters main part of the asymptotic solution of this homogenized equation may contain additional dispersion corrections.

1 Introduction

We consider the following Cauchy problem

$$\frac{\partial^2}{\partial t^2} u(x,\, t) = \mathrm{C}^2\left(\frac{\Theta(x)}{\varepsilon},\, x\right) \Delta u(x,\, t),\; x \in \mathbb{R}^n, \qquad u|_{t=0} = V\left(\frac{x}{\mu}\right),\; u_t|_{t=0} = 0. \tag{1}$$

Here the vector function $\Theta(x) = (\theta_1(x),\, \dots,\, \theta_m(x))$, $m \leq n$ describes the form of the fast oscillations. Functions $\theta_k(x)$ are smooth real-valued functions and their gradients $\nabla\theta_k(x)$ are linealry independent for all x. In various works the case $\Theta(x) = x$ usually is considered.

Function $\mathrm{C}^2(y,\, x)$, $y \in \mathbb{R}^m$ is 2π-periodical with respect to each variable y_j and smooth with respect to all variables. In addition this function is bounded

$$0 < c_m \leq \mathrm{C}^2(y,\, x) \leq c_M.$$

Small parameter $0 < \varepsilon \ll 1$ describes the length of the fast oscillations. The small parameter $0 < \mu \ll 1$ describes the length of the propagating wave. We assume that the relation between those parameters is $\varepsilon \leq \mu^{3/2}$.

The localization of the initial perturbation leads to the localization of the propagating wave and it is localized near some hypersurface at each time moment. Therefore it is possible to use the semiclassical approximation and to construct the analytic asymptotic formulas which give small residue for (1).

In order to implement the semiclassical analysis one has to reduce the initial equation to the equation with smooth coefficients which we call the homogenized equation. In order to obtain such equation we use the adiabatic approximation in the operator form and separating fast and slow variable.

It turns out that when $\varepsilon \ll \mu^{3/2}$ then the asymptotic solution is described by the wave equation with smooth coefficient. On the other hand if $\varepsilon = \mu^{3/2}$ then the dispersion effects appear in the asymptotic solution because the presence of the correction to the wave equation.

In present work we will focus ourselfs to the part of the homogenization procedure.

2 Main Results

We are looking for the solution of the problem (1) in the form

$$u(x,\,t)=\Psi\left(\frac{\Theta(x)}{\varepsilon},\,x,\,t\right),$$

where function $\Psi(y,\,x,\,t)$ is also 2π-periodic with respect to the variables $y\in\mathbb{R}^m$.

Substitution of this form into the equation (1) leads to the equation for the function $\Psi(y,\,x,\,t)$

$$-\varepsilon^2\Psi_{tt}(y,\,x,\,t)=\mathrm{C}^2(y,\,x)\Big\langle(-i\varepsilon\nabla-i\nabla_y^\theta),\,(-i\varepsilon\nabla-i\nabla_y^\theta)\Big\rangle\Psi(y,\,x,\,t),\tag{2}$$

where $-i\nabla_y^\theta=-i(\nabla\Theta(x))\nabla_y$, matrix $\nabla\Theta(x)=(\nabla\theta_1(x),\,\dots,\,\nabla\theta_m(x))$, and $\nabla_y=(\partial/\partial y_1,\,\dots,\,\partial/\partial y_m)$ and the brackets stand for the Euclidean vector inner product.

It is convenient to use the differential operator in the form $-i\mu\nabla$, since the localization parameter is μ and the ordinary derivative is of order $O(1/\mu)$.

Solution of the equation (2) is looking for in the form of some pseudo-differential operator $\hat\chi$ acting on some function $v(x,\,t)$: $\quad\Psi(y,\,x,\,t)=\hat\chi(\overset{2}{x},\,-i\overset{1}{\mu\nabla},\,y;\,\varepsilon,\,\mu)v(x,\,t)$.
Here digits show the order of action: first differentiation and second — multiplication on the variable.

The main assumption is function $v(x,\,t)$ does not depend on variables y and moreover satisfies the homogenized equation with smooth coefficients

$$-\mu^2v_{tt}(x,\,t)=\frac{\mu^2}{\varepsilon^2}\hat L(\overset{2}{x},\,-i\overset{1}{\mu\nabla};\,\varepsilon,\,\mu)v(x,\,t).$$

Theorem 2.1. *The symbol $L(x,\,p;\,\varepsilon,\,\mu)$ of the operator $\hat L$ has the following expansion with respect to ε/μ*

$$\frac{\mu^2}{\varepsilon^2}L(x,\,p;\,\varepsilon,\,\mu)=c^2(x)|p|^2-\frac{\varepsilon^2}{\mu^2}c^2(x)\Phi(x)|p|^4+\frac{\varepsilon^4}{\mu^4}\mathcal{L}(x,\,p,\,\frac{\varepsilon}{\mu}).\tag{3}$$

Function $c(x)$ is defined with the help of the following average

$$c^2(x)=\left\langle\frac{1}{\mathrm{C}^2(y,\,x)}\right\rangle_{\mathbb{T}^m}^{-1}=\left(\frac{1}{(2\pi)^m}\int\limits_{\mathbb{T}^m}\frac{dy}{\mathrm{C}^2(y,\,x)}\right)^{-1}.$$

Function $\Phi(x)=\langle|\nabla_y^\theta\psi_2(y,\,x)|^2\rangle_{\mathbb{T}^m}$, where $\psi_2(y,\,x)$ is the solution of the cell problem

$$(-\Delta_y^\theta)\psi_2=\frac{c^2(x)-\mathrm{C}^2(y,\,x)}{\mathrm{C}^2(y,\,x)},\qquad\langle\psi_2(y,\,x)\rangle_{\mathbb{T}^m}=0.\tag{4}$$

Here $(-\Delta_y^\theta)=\langle-i\nabla_y^\theta,\,-i\nabla_y^\theta\rangle$. Coerrection $\mathcal{L}$ contains the derivatives of the higher (copmpare to $|p|^4$) order.

Conclusion. We need to determine the homogenized equation only with the terms of order $O(\mu)$ because we use the semiclasscal approximation and the higher terms do not play role in the asymptotic formulas. In case $\varepsilon=\mu^{3/2}$ we need to take into account correction $c^2(x)\Phi(x)|p|^4$ which leads to the appearance of the dispersion effects in the asymptotic solution. In that case the ratio $\varepsilon^4/\mu^4=\mu^2$ and the correction $\mathcal{L}$ in (3) to the symbol does not play any role in the construction of the semiclassical asymptotics.

On the other hand it can be seen from (3), that if $\varepsilon\ll\mu^{3/2}$, then the correction $c^2(x)\Phi(x)|p|^4$ does not play role, since the ratio ε^2/μ^2 is smaller than μ. In that case we obtain the wave equation without any dispersion effects for the asymptotics. In that case we also do not need to solve the cell problem (4) because the solution of this problem enters in the dispersion correction.

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HOMOGENIZATION OF THE WAVE EQUATION IN THE NON-DIVERGENT FORM IN THE WHOLE SPACE WITH INHOMOGENIOUS MEDIA

M. NEKLYUDOV[1], W. NEVES[2], S. SERGEEV[3] & E. SHAMAROVA[4]

[1]Universidade Federal das Amazonas, Brazil, misha.neklyudov@gmail.com
[2]Universidade Federal do Rio de Janeiro, Brazil, wladimir@im.ufrj.br
[3]Pontifícia Universidade Catolico do Rio de Janeiro, Brazil, sergeevse1@gmail.com
[4]Universidade Federal da Paraíba, Brazil, evelina.shamarova@academico.ufpb.br

Abstract

In many physical applications the wave equation appears in the non-divergent form. If the media is inhomogeneous then the coefficients of such equation are variable and in various situations the media may contain some fast-oscillating fluctuations. In the later case the homogenization problem can be posed.

We are considering the homogenization problem of the waves propagation in the whole space with fast oscillating bounded coefficients.

Our main results are on the existence of the solution of the Cauchy problem for a wave equation and the homogenization procedure based on the 2-scale convergence.

1 Introduction

We consider the following Cauchy problem for the inhomogeous wave equation

$$\frac{1}{c_\varepsilon^2(x)}u_{tt}(x,\,t) - \Delta u(x,\,t) = U(x)f(t), \quad x \in \mathbb{R}^2, \tag{1}$$

$$u|_{t=0} = 0,\, u_t|_{t=0} = 0. \tag{2}$$

Function $c_\varepsilon(x) \equiv c(x,\,\Theta(x)/\varepsilon)$ describes the velocity of the waves in the media where function $\Theta(x)$ describes the fast-oscillating fluctuations of the media. The small parameter $\varepsilon \ll 1$ describes the scale of these fast oscillations.

We assume that the mapping $\Theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a bi–Lipschitz diffeomorphism. We assume also that there are two positive constants m and M such that

$$\operatorname*{ess\,inf}_{y\in\mathbb{R}^2}\big(|\big(\nabla\Theta(y)\big)|\big) \geq m > 0, \quad \operatorname*{ess\,sup}_{y\in\mathbb{R}^2}\big(|\nabla\Theta(y)|\big) \leq M < \infty.$$

As for the function $c(x,\,y)$, we assume that this function is continuous, bounded

$$0 < c_{min} \leq c(x,\,y) \leq c_{max} < +\infty$$

and periodic with respect to variables y with period 1.

The right-hand side of the (1) describes the source which generates the waves. In the spacial coordinates this source has the form of the function $U(x)$ and we assume that this function belongs to the $L_2(\mathbb{R}^2)$. The action of the source in time is described by the function $f(t)$. The description of the function $f(t)$ is better to present via another function $g_0(\lambda t)$, where $f(t) = \lambda^2 g_0'(\lambda t)$ and

$$g_0(0) = 0, \quad \int_0^{+\infty} g_0(\tau)d\tau = 1, \quad |g_0^{(k)}(t)| \leq Ce^{-\nu t},\, k = 0,\, 1,\, 2,\, \ldots, \quad \nu > 0.$$

Here the constant λ is the inverse to the time value of action of the source in time.

2 Main results

The problem (1), (2) in the present statement is very important from the case of the physical applications. The main assumption of only boundness of the coefficient $c(x, \Theta(x)/\varepsilon)$ is also very important, because the trivial example of such coefficient is the constant c which is only bounded in the whole space $\mathbb{R}^2$. From the other hand we cannot assume, in general, the smoothness of the coefficient.

It turns out that in the present formulation with mentioned above conditions the problem (1), (2) was not very well studied. Usually some smoothnes of the coefficients together with dependence only on the oscillations $\Theta(x)/\varepsilon$ was assumed, see for example [5, 3, 1].

We begin with the existence theorem for the problem (1), (2). For that we reduce this problem to the system of the equation in the weak of Schrödinger form. Let us introduce the functional space $\mathcal{H} \equiv L_{2,\,c}(\mathbb{R}^2; \mathbb{C}) \oplus L_2(\mathbb{R}^2;\, \mathbb{C}^2)$ with the following inner product. For any $\Psi = (u,\, v)^T$ and $\Psi = (f,\, g)^T$ from $\mathcal{H}$ we have

$$(\Psi,\, \Phi)_{\mathcal{H}} = \left(\frac{1}{c_\varepsilon^2(x)} u(x),\, f(x)\right)_{L_2(\mathbb{R}^2)} + \int\limits_{\mathbb{R}^2} \langle v(x),\, \overline{g(x)}\rangle dx,$$

where $\langle \cdot,\, \cdot\rangle$ denotes the Eclidean inner product of the vectors.

Let us define the following function $\mathcal{F} = (F(x,\, t),\, 0) \in \mathcal{H}$, where $F(x,\, t) = i\lambda U(x) g_0(\lambda t)$ and let us introduce the function $\Psi_\varepsilon = (u_\varepsilon,\, v_\varepsilon)$, where u_ε is the solution of the initial problem (1), (2) and vector-function v_ε can be viewed as a gradient of u_ε.

Definition 2.1. *For given $T > 0$, function $\Psi_\varepsilon(t) \in C([0,\, T],\, \mathcal{H})$ is the weak solution of the equation (1), (2), if for any function $\Phi \in \mathcal{H}$ the following equality holds*

$$i\frac{d}{dt}(\Psi_\varepsilon(t),\, \Phi)_{\mathcal{H}} = (\Psi_\varepsilon(t),\, \hat{W}_\varepsilon \Phi)_{\mathcal{H}} + (c_\varepsilon^2(x)\mathcal{F}(t),\, \Phi)_{\mathcal{H}}. \tag{3}$$

Here the operator

$$\hat{W}_\varepsilon := \begin{pmatrix} 0 & c_\varepsilon^2(x)(-i\,\mathrm{div}) \\ -i\nabla & 0 \end{pmatrix} \tag{4}$$

is the self-adjoined operator in $\mathcal{H}$.

From [4], we can conclude that there exists unique weak solution of the equation (3).

Following the 2-scale convergence [3] and its θ-modification [1] we provide the homogenization procedure when $\varepsilon \to 0$ and derive the homogenized equation.

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THE STUDY OF COMPRESSIBLE EULER EQUATIONS IN TWO SPATIAL VARIABLES FOR NON-POTENTIAL FLOWS, AND AN ANALYSIS OF THE PSEUDO-SUBSONIC REGIME IN THE POTENTIAL CASE

L. BONALDO[1], T. MELLO[1] & W. NEVES[1]

[1]Universidade Federal do Rio de Janeiro, IM, RJ, Brazil
nn.lauren@gmail.com, souzamello.talita@gmail.com, wladimir@im.ufrj.br

Abstract

In this research we consider the compressible, isentropic, Euler equations in two spatial variables for a generalized polytropic gas law which encompass, in particular, the Chaplygin gas. Our first objective was to use the Hodge-Helmholtz decomposition to obtain a Bernoulli-type equation and a system satisfied by the pseudo-velocity without assuming irrotational flows. After, under the assumption of self-similar potential flows, we have been shown that the Euler equations obey an ellipticity principle, which means roughly speaking that, a pseudo-supersonic bubble cannot form within a pseudo-subsonic region during a continuous variation of the gas flow. Finally, we use the ellipticity principle to prove the existence of solutions in the pseudo-subsonic region, which is modeled by an elliptic degenerated equation.

1 Introduction

Let us consider the isentropic Euler equations in two space dimensions given by

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho \mathbf{v}) = 0 \quad \text{(continuity equation)}, \\ \partial_t(\rho \mathbf{v}) + \nabla_x \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_x p = 0 \quad \text{(linear momentum equation)}. \end{cases} \tag{1}$$

Here $t \in (0, +\infty)$ is the time and $x = (x_1, x_2) \in \mathbb{R}^2$ the space coordinates, $\rho = \rho(t, x) > 0$ denotes the density, $\mathbf{v} = \mathbf{v}(t, x) = (v_1, v_2)$ the velocity field, $p = p(\rho) \in \mathbb{R}$ is the pressure law. We will suppose $p'(\rho) > 0$ and denote $c^2 := p'(\rho)$, where c is called the sound speed. We define the enthalpy $h(\rho)$ as $h'(\rho) = \frac{p'(\rho)}{\rho} > 0$. Also, we recall that $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product of vectors $\mathbf{a}$ and $\mathbf{b}$, that is, $\mathbf{a} \otimes \mathbf{b} = [a_i b_j]_{ij} \quad (i, j = 1, 2)$. We consider self-similar solutions, which depend only on the similarity coordinates, denoting $(\mathbf{v}, p, \rho)(t, x_1, x_2) = (\mathbf{v}, p, \rho)(\xi_1, \xi_2)$, $\xi = (\xi_1, \xi_2) = \frac{x}{t} = \left(\frac{x_1}{t}, \frac{x_2}{t}\right)$. Also, we define $U(\xi_1, \xi_2) \equiv (U^1, U^2)(\xi_1, \xi_2) := \mathbf{v}(\xi_1, \xi_2) - (\xi_1, \xi_2)$, called the pseudo-velocity. Under the Hodge-Helmholtz decomposition, there exists ψ, W such that $U = (U^1, U^2) = \nabla \psi + W$, $\operatorname{div} W = 0$. Then, the flow will be rotational or irrotational (potential) if $W \neq 0$ or $W = 0$, respectively. We also denote $U^\perp := (-U^2, U^1)$ and $\omega := \operatorname{rot} U = \frac{\partial U^2}{\partial \xi_1} - \frac{\partial U^1}{\partial \xi_2}$, where ω is called the vorticity of the fluid. We notice that, under the Hodge-Helmholtz decomposition, $\omega = \operatorname{rot} U = \operatorname{rot} W$. We say that the flow is pseudo-subsonic, pseudo-sonic or pseudo-supersonic at (t, x) if $|U| < c(\rho(t, x))$, $|U| = c(\rho(t, x))$ or $|U| > c(\rho(t, x))$, respectively. We define the pseudo-Mach-number as $L = \frac{|U|}{c}$. Then, the flow is pseudo-subsonic, pseudo-sonic or pseudo-supersonic at (t, x) if $L < 1$, $L = 1$ or $L > 1$, respectively. We fix the pressure law $p = p(\rho)$ as follows:

$$p(\rho) = \frac{a^2}{\gamma}(\rho^\gamma - \underline{\rho}^\gamma), \tag{2}$$

where $a > 0$, $\gamma \in [-1, +\infty) \backslash \{0\}$, $\rho > \underline{\rho} \geq 0$. We notice that for $\gamma > 1$ we have the polytropic gas and for $\gamma = -1$ we have the Chaplygin gas.

2 Main Results

Theorem 2.1. *Suppose that* $(\mathbf{v}, p, \rho)(t, x)$ *satisfies the isentropic Euler equations and that* U*, the pseudo-velocity, is a* C^2 *function.*

1. *Then, there exists* $\mathcal{F}$ *such that* $c^2 = \mathcal{F}(U)$ *and the pseudo-velocity* U *satisfies the system*

$$\begin{cases} c^2 \operatorname{div} U - (DU)U \cdot U = |U|^2 - 2c^2, \\ \operatorname{div}(\omega U) + \omega = 0. \end{cases} \tag{1}$$

2. *Under the Hodge-Helmholtz decomposition, if* $U = \nabla\psi + W$*, with* $\operatorname{div} W = 0$*, there exists* F *such that* $\nabla F = -\omega U^{\perp} - W$*. We also have that* $c^2(U)$ *is given by*

$$c^2(U) = (\gamma - 1)\left(F - \psi - \frac{1}{2}|U|^2\right) + C, \quad C \in \mathbb{R}, \tag{2}$$

if $\gamma \neq 1$*, and* $c^2 = a^2$*, if* $\gamma = 1$*. Also, holds the Bernoulli type equation*

$$h + \psi + \frac{1}{2}|U|^2 = F + C, \quad C \in \mathbb{R}. \tag{3}$$

Now, suppose $U = \nabla\varphi$. In this case, system (1) reduces to

$$c^2 \Delta\varphi - \sum_{i,j=1}^{2} \varphi_i \varphi_j \varphi_{ij} = |\nabla\varphi|^2 - 2c^2, \tag{4}$$

where $c^2(\varphi) = -(\gamma - 1)\left(\varphi + \frac{1}{2}|\nabla\varphi|^2\right) + C$, $C \in \mathbb{R}$ if $\gamma \neq 1$ and $c^2 = a^2$ if $\gamma = 1$.

Theorem 2.2. *(Ellipticity principle). Let* $\Omega \subset \mathbb{R}^2$ *be an open bounded domain and suppose* $\gamma > -1$*.*

(i) Let $\varphi \in C^3(\Omega)$ *satisfy (4) with* $L \leq 1$ *and* $\rho > 0$ *in* Ω*. Then either* $L \equiv 0$ *in* Ω *or* L *does not attain its maximum in* Ω*.*

(ii) For any $D > 0$*, there exists* $C_0 > 0$ *depending only on* (γ, D) *such that, if* $\operatorname{diam}(\Omega) \leq D$*, for any* $\delta \geq 0$*,* $\hat{c} \geq 1$*, and* $b \in C^2(\Omega)$ *with* $|Db| + \hat{c}|D^2 b| \leq \frac{\delta}{\hat{c}}$*, and for any solution* $\varphi \in C^3(\Omega)$ *of (4) satisfying* $L \leq 1$*,* $\rho(|D\varphi|^2, \varphi) > 0$*, and* $c(|D\varphi|^2, \varphi) \leq \hat{c}$ *in* Ω*, then either* $L^2 \leq C_0\delta$ *in* Ω *or* $L^2 + b$ *does not attain its maximum in* Ω*.*

Theorem 2.3. *If* $\gamma > 1$*, the Dirichlet problem*

$$\begin{cases} c^2 \Delta\varphi - \sum_{i,j=1}^{2} \varphi_i \varphi_j \varphi_{ij} = |\nabla\varphi|^2 - 2c^2, \quad in \quad \Omega; \\ \varphi = \varphi_0 \quad on \quad \partial\Omega, \end{cases} \tag{5}$$

has a solution $\varphi \in C^3(\Omega) \cap C^1(\overline{\Omega})$*, where* Ω *is the subsonic region given by* $L < 1$*,* $\partial\Omega \in C^{3,\alpha}$*,* $\varphi_0 \in C^{3,\alpha}(\overline{\Omega})$*.*

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MEAN FIRST-PASSAGE TIME IN DIFFUSION WITH STOCHASTIC RESETTING WITH ALTERNATING BOUNDARIES

T. T. DA SILVA

DEFIM, Universidade Federal de São João Del Rei, MG, Brazil, timoteo@ufsj.edu.br

Abstract

We develop a model of diffusion with stochastic resetting with alternating boundaries. We study the first-passage time and find that its mean can be minimized with respect to the resetting rate. However the model allows for multiple extrema in contrast to the simple diffusion with stochastic resetting [3].

1 Introduction

Consider a particle performing a diffusive motion according to a standard one-dimensional Brownian motion W_t starting at the origin, such that $\mathbb{E}[W_t] = 0$ and $\mathbb{E}[W_t^2] = t$. Also let $c(t)$ be a C^1 curve on $(0, \infty)$. Let $\tau_{W,c} = \inf\{t > 0 : W_t \geq c(t)\}$ be the random variable (RV) indicating the first-passage time (FPT) of W_t through $c(t)$. If $f(t)$ is the probability density function of $\tau_{W,c}$, then it is known that it satisfies the following Volterra integral equation [5]

$$\Psi\left(\frac{c(t)}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{c(t)-c(s)}{\sqrt{t-s}}\right) f(s)ds,\ t > 0, \tag{1}$$

where $\Psi(x) = \int_x^\infty \phi(z)dz$ and $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. The solution when $c(t) = at+b$, $b > 0$, can be computed explicitly [2, 4] and reads $f(t) = \frac{b}{t^{3/2}}\phi\left(\frac{at+b}{\sqrt{t}}\right)$. In particular, when $a = 0$, $\tau_{W,c}$ follows a Lévy distribution [1].

The boundary $c(t) = at + b$ leads $\tau_{W,c}$ to have infinite mean if $a \geq 0$. This is inconvenient, but it can be overcome. One way to obtain finite mean FPT (MFPT) is to change the distribution of the process by introducing resettings [3]. Consider the case when the boundary is given by $c_1(t) = x_0$. Evans & Majumdar discovered, in their seminal work, that resetting the diffusion at rate β can lead to a finite MFPT, which is given by the expression

$$\mathbb{E}[\tau_{W,x_0,\beta}] = \frac{e^{x_0\sqrt{2\beta}} - 1}{\beta} \tag{2}$$

Moreover, the MFPT can be minimized with respect to β and the optimal β^* is approximately $1.2698/x_0^2$.

In this work we modify the scheme introduced by Evans & Majumdar and allow the boundary to alternate between two curves, $c_1(t) = x_0$ and $c_2(t) = at + b$, in such a way that the process develops as follows:
Step 1: The particle starts at the origin and diffuses until either reaching the boundary $c_1(t) = x_0$ or resetting to the origin (go to Step 2);
Step 2: If it resets, then it resumes the diffusion until reaching the boundary $c_2(t - T) = a(t - T) + b$ (here T is the time of the last reset) or resetting to the origin (go to Step 3);
Step 3: If it resets, we move to Step 1 again;
Step 4: The process repeats itself until the particle is finally absorbed by the alternating boundary.
Figure 1(a) shows an illustrative sample path for this process.

2 Main Result

Theorem 2.1. *Let W_t be a standard one-dimensional Brownian motion with $W_0 = 0$. Let τ_i be a sequence of independent and identically distributed exponential RVs with rate β, independent of W_t. Write $T_j = \sum_{i=1}^j \tau_i$. Fix*

(a)

(b)

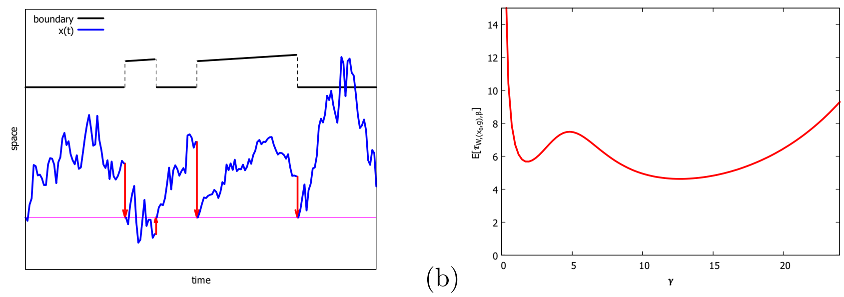

Figure 1: In part (a), process $X(t)$ is a diffusion between the resets indicated by the red arrows. The boundary alternates between two curves. It is depicted the first passage of $X(t)$ through boundary c_1; passages of other orders are also depicted. Part (b) corresponds to the MFPT with respect to γ, in the case where $x_0 = 1$, $a = 10$, $b = 0.2$.

$x_0, b > 0$ and $g(t) = at + b$, where $a \in \mathbb{R}$. Define the process X_t by $X_t = W_t - W_{T_j}$, if $t \in [T_j, T_{j+1})$. For $n \geq 0$, consider the sets $S_{2n} = \{t > 0 : X_t \geq x_0, t \in [T_{2n}, T_{2n+1})\}$, $S_{2n+1} = \{t > 0 : X_t \geq g(t - T_{2n+1}), t \in [T_{2n+1}, T_{2n+2})\}$ and $S = \cup S_n$. Then the RV $\tau_{W,(x_0,g),\beta} = \inf S$ has finite mean given by

$$\mathbb{E}[\tau_{W,(x_0,g),\beta}] = \frac{\left(e^{x_0\sqrt{2\beta}} - 1\right)\left[2e^{b(a+\sqrt{2\beta+a^2})} - 1\right]}{\beta\left[e^{x_0\sqrt{2\beta}} + e^{b(a+\sqrt{2\beta+a^2})} - 1\right]}. \quad (1)$$

The proof relies on a probabilistic reasoning to establish an integral equation for the survival probability $S(t) = \mathbb{P}(\tau_{W,(x_0,g),\beta} > t)$, which is then solved in Laplace space. The remarkable consequence of expression (1) is that $\mathbb{E}[\tau_{W,(x_0,g),\beta}]$ exhibits multiple extrema. In Fig. 1(b), it is shown an illustrative example of the variation of $\mathbb{E}[\tau_{W,(x_0,g),\beta}]$ with regard to $\gamma = x_0\sqrt{2\beta}$.

3 Acknowledgements

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XVIII ENAMA - Novembro 2025 *153–154*

DYNAMICS FOR A PLATE MODEL WITH DEGENERATE NONLOCAL STRONG ENERGY DAMPING

VANDO NARCISO

Center of Exact and Earth Sciences, UEMS, MS, Brasil, vnarciso@uems.br
Financiado por Fundect/CNPQ 15/2024.

Abstract

This paper is devoted to initial-boundary value problem of a beam/plate equation with nonlocal damping that is derived from dissipative energy models for flight structures proposed by Balakrishnan-Taylor [2]. We prove the global existence and uniqueness of weak solutions. Furthermore, we prove that the dynamic system generated by the weak solutions of the problem has a compact global attractor.

1 Introduction

In this paper, we study the following beam/plate equation with degenerate nonlocal energy damping

$$\partial_t^2 u + \Delta^2 u - \gamma E(u, \partial_t u)^q \Delta \partial_t u + f(u) = h \quad \text{in} \quad \Omega \times \mathbb{R}^+. \tag{1}$$

where $E(u, \partial_t u) = \frac{\|\Delta u(t)\|^2 + \|\partial_t u(t)\|^2}{2}$ is the energy associated with the linear part of the system, $\gamma > 0, q \geq 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain of with smooth boundary $\Gamma = \partial\Omega$, f is a nonlinear function like $f(u) \approx |u|^\rho u - C$, h is an external force, and $\|\cdot\|$ stands for the norm in $L^2(\Omega)$. We consider either clamped or hinged boundary conditions, described respectively by

$$u|_{\Gamma \times \mathbb{R}^+} = \frac{\partial u}{\partial \nu}|_{\Gamma \times \mathbb{R}^+} = 0 \quad \text{or} \quad u|_{\Gamma \times \mathbb{R}^+} = \Delta u|_{\Gamma \times \mathbb{R}^+} = 0, \tag{2}$$

where ν is the unit exterior normal to Γ. The initial conditions associated to (1) are given by

$$u(x, 0) = u_0(x) \quad \text{and} \quad \partial_t u(x, 0) = u_1(x), \quad x \in \Omega. \tag{3}$$

Let us denote $W_0 = L^2(\Omega)$ and $W_1 = H_0^1(\Omega)$, and to attend the two boundary conditions in (2) we define $W_2 = H_0^2(\Omega)$ or $W_2 = H^2(\Omega) \cap H_0^1(\Omega)$. Let $\lambda_1 > 0$ the first eigenvalue of the bi-harmonic operator Δ^2 in W_2.

Assumption 1.1. *The external force $h \in W_0$ and f is a C^1-function on $\mathbb{R}$ satisfying*

A1. $|f'(s)| \leq C_{f'}(1 + |s|^\rho), \quad \forall\, s \in \mathbb{R}$,

A2. $-C_f - \frac{\alpha}{2}s^2 \leq \hat{f}(s) := \int_0^s f(\tau)d\tau \leq f(s)s + \frac{\alpha}{2}s^2, \quad \forall\, s \in \mathbb{R}$,

where we consider $C_{f'} > 0$, $C_f \geq 0$, $0 \leq \alpha < \lambda_1$, and $\rho > 0$ if $1 \leq n \leq 4$ or $0 < \rho \leq \frac{4}{n-4}$ if $n \geq 5$;

Our analysis with respect to the global existence and long-time behavior of solutions is given on the phase space $\mathcal{H} = W_2 \times W_0$ equipped with norm $\|(u, v)\|_{\mathcal{H}}^2 = \|\Delta u\|^2 + \|v\|^2$.

2 Mathematical Results

The existence and uniqueness results of the global weak solutions in the space $\mathcal{H}$ are given in the following theorem.

Theorem 2.1. *Let $T > 0$ be arbitrary, $\gamma > 0$, and $q \geq 1$. Under Assumption 1.1 we have: if initial data $(u_0, u_1) \in \mathcal{H}$, then problem (1)-(3) has a unique weak solution*

$$(u, \partial_t u) \in C([0, T], \mathcal{H}), \quad \forall\, T > 0, \tag{4}$$

satisfying

$$u \in L^\infty(0, T; W_2), \quad \partial_t u \in L^\infty(0, T; W_0) \quad \text{and} \quad \partial_t^2 u \in L^2(0, T; W_2'). \tag{5}$$

Proof The principle of the proof is classical. We using the Faedo-Galerkin method associated to compactness arguments. The well-posedness of problem (1)-(3) given by Theorem 2.1 implies that the evolution operator $S_t : \mathcal{H} \to \mathcal{H}$ defined by

$$S_t(u_0, u_1) = (u(t), \partial_t u(t)), \quad t \geq 0, \tag{6}$$

where (u, u_t) is the unique weak solution of the system (1)-(3), defines a nonlinear C_0-semigroup which is locally Lipschitz continuous on the phase space $\mathcal{H}$. Therewith the dynamics of problem (1)-(3) can be studied through the continuous dynamical system $(\mathcal{H}, S_t)$.

Our main result in the present work is the following.

Theorem 2.2. *Assume that hypotheses of Theorem 2.1 hold. Then, the associate dynamical system $(\mathcal{H}, S_t)$ of problem (1)-(3) has a compact global attractor $\mathfrak{A}$ in $\mathcal{H}$, which is characterized by the unstable manifold $\mathfrak{A} = \mathrm{M}^u(\mathcal{N})$ emanating from the set of stationary solution $\mathcal{N}$. In addition, $\mathfrak{A}$ consist of full trajectories $\Upsilon = \{U(t) = (u(t), u_t(t)) : t \in \mathbb{R}\}$ such that*

$$\lim_{t \to -\infty} dist_{\mathcal{H}}(U(t), \mathcal{N}) = 0 \quad \text{and} \quad \lim_{t \to +\infty} dist_{\mathcal{H}}(U(t), \mathcal{N}) = 0.$$

Proof The existence of a compact global attractor is granted once our dynamical system $(\mathcal{H}, S_t)$ is **gradient**, **dissipative**, and that the operators $\{S_t\}$ are **uniformly compact** in $\mathcal{H}$ for t large. Then by direct application of [5, Theorem 1.1, Chapter I] and [4, Theorem 7.5.6, Chapter 7] we obtain the proof of the Theorem 2.2. □

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ON THE EXISTENCE OF STATIONARY VORTEX PATCHES FOR THE GSQG IN BOUNDED DOMAINS

E. CUBA[1], V. ANGULO-CASTILLO[2] & L. C. F. FERREIRA[2]

[1]Universidad Nacional de Colombia, Sede Orinoquia, Colombia, edisonfausto@gmail.com

[2]Universidade Estadual de Campinas, IMECC, Brazil, vlcastillo@unal.edu.co, lcff@ime.unicamp.br

Abstract

In this congress, we present a result regarding the existence of time-periodic vortex patches for the generalized surface quasi-geostrophic (SQG) equation within a bounded domain. This result is established for values of γ in the range $(1, 2)$. The vortex patches obtained have fixed vorticity and total flux and are situated near non-degenerate critical points of the Kirchhoff-Routh equation. The proof is achieved by analyzing the linearization of the contour dynamics equation and applying the implicit function theorem, using carefully chosen function spaces.

1 Introduction

In this study, we delve into a category of active scalar systems that interact with an incompressible flow within a two-dimensional framework. To be precise, through an examination of the contour dynamics equation and the application of the implicit function theorem, we establish the presence of stationary vortex patches. These patches are characterized by a constant total flux and a fixed vorticity for each individual patch within the framework of the generalized surface quasi-geostrophic (gSQG) equations, which are defined within a bounded domain. In that sense, this model explains how the potential temperature ω evolves under the influence of the transport equation

$$\begin{cases} \partial_t \omega + \mathbf{v} \cdot \nabla \omega = 0 & \text{in } \Omega \times (0, T), \\ \mathbf{v} = \nabla^{\perp} (-\Delta)^{-1+\frac{\gamma}{2}} \omega & \text{in } \Omega \times (0, T), \\ \omega\big|_{t=0} = \omega_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in two-dimensional space, and we consider a parameter γ satisfying the condition $0 \leq \gamma < 2$. The variable $\omega(\boldsymbol{x}, t)$, defined for $\boldsymbol{x}$ within Ω and t in the interval $(0, T)$, represents an active scalar being advected by a velocity field $\mathbf{v}(\boldsymbol{x}, t)$. This velocity field is generated by ω, and $\nabla^{\perp} = (\partial_2, -\partial_1)$. The operator denoted as $(-\Delta)^{-1+\frac{\gamma}{2}}$ is defined by $(-\Delta)^{-1+\frac{\gamma}{2}} \omega(\boldsymbol{x}) = \int_{\Omega} K_{\gamma}(\boldsymbol{x}, \boldsymbol{y}) \omega(\boldsymbol{y}), d\boldsymbol{y}$, where the term $K_{\gamma}(\boldsymbol{x}, \boldsymbol{y})$ represents the Green function associated with the fractional Laplacian in bounded domains with smooth boundaries $(-\Delta)^{-1+\frac{\gamma}{2}}$. It is defined for each pair of points $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, where $\boldsymbol{x} \neq \boldsymbol{y}$, as follows:

$$K_{\gamma}(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} -\frac{1}{2\pi} \log |\boldsymbol{x} - \boldsymbol{y}| + K_0^0(\boldsymbol{x}, \boldsymbol{y}), & \gamma = 0, \\ \frac{C_{\gamma}}{|\boldsymbol{x} - \boldsymbol{y}|^{\gamma}} + K_{\gamma}^0(\boldsymbol{x}, \boldsymbol{y}), & \gamma \in (0, 2), \end{cases} \tag{2}$$

with $\Gamma(\cdot)$ being the Euler gamma function and $C_{\gamma} = \frac{2^{\gamma-1}\Gamma(\frac{\gamma}{2})}{\Gamma(1-\frac{\gamma}{2})}$. Additionally, K_{γ}^0 belongs to the class of infinitely differentiable functions $C^{\infty}(\Omega \times \Omega)$, as discussed in Lemma 2.3 of [1].

We highlight that the system (1) covers the cases of the 2D incompressible Euler equations by taking $\gamma = 0$ and the inviscid SQG equations when considering $\gamma = 1$. The system (1) for $0 < \gamma < 2$ was initially introduced by Córdoba *et al.* for the flat case $\mathbb{R}^2$ in their work [2]. Over the past decade, it has garnered significant attention and scruting as it represents a generalization of both the Euler equation and the SQG equation. Notice that the case

$\gamma = 2$ produces stationary solutions. The study of SQG-type equations defined on bounded smooth domains is, in part, much more complicated than in the $\mathbb{R}^2$ case, since there the Green function cannot be expressed explicitly. A key point in the development of our problem (1) is the analysis of the problem of desingularization of point vortices, which is related to the search for concentrated global solutions as performed in [3].

For a collection of n real numbers $\kappa_1, \kappa_2, \ldots, \kappa_n$, we establish the Kirchhoff-Routh function on Ω^n in the following manner

$$\mathcal{W}_n(x_1, x_2, .., x_n) = -\sum_{i \neq j}^{n} \kappa_i \kappa_j K_\gamma^1(x_i, x_j) + \sum_{i=1}^{k} \kappa_i^2 K_\gamma^0(x_i, x_i), \tag{3}$$

where Ω^n is the set of vectors $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ such that each x_i belongs to the set Ω for $i = 1, 2, \ldots, n$ and $K_\gamma^1(x, y) = \frac{C_\gamma}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma}$.

2 Main Result

Our main result is written as follows:

Theorem 2.1. *Consider a bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary and m given positive values κ_i $(i = 1, \ldots, n)$. Assume that $\boldsymbol{x}_0 = (x_{0,1}, \ldots, x_{0,n}) \in \Omega^n$ with $x_{0,i} \neq x_{0,j}$ for $i \neq j$ is an isolated critical point of $\mathcal{W}_m$ as defined in (3) and it fullfills satisfying the nondegeneracy condition: deg $(\nabla \mathcal{W}_n, \boldsymbol{x}_0) \neq 0$. Under these conditions, there exists $\varepsilon_0 > 0$, such that for all $0 < \varepsilon < \varepsilon_0$, a stationary vortex patch solution ω_ε can be constructed, which exhibits the following characteristics:*

(i) $\omega_\varepsilon = \sum_{i=1}^n \frac{1}{\varepsilon^2} \chi_{\Gamma_i}$ *within specific domains* $\Gamma_i \subset \Omega, i = 1, \ldots, n$.

(ii) *The boundaries $\partial\Gamma_i$ for $i = 1, \ldots, n$ can be defined using the subsequent parameterization*

$$\partial\Gamma_i = \left\{ x_{\varepsilon,i} + \varepsilon \left(\sqrt{\frac{\kappa_i}{\pi}} + o(1) \right) (\cos\beta, \sin\beta) \mid \beta \in [0, 2\pi) \right\}, \tag{4}$$

where $x_{\varepsilon,i} = x_{0,i} + o(1)$ as $\varepsilon \to 0$.

(iii) *The total flux for each patch remains fixed as $\frac{1}{\varepsilon^2} |\Gamma_i| = \kappa_i$, $\forall i = 1, \ldots, m$.*

(iv) *As $\varepsilon \to 0^+$, one has the following convergence in the sense of measure $\omega_\varepsilon \to \sum_{i=1}^n \delta(x - x_{0,i})$ weakly, where $\delta(x - x_{0,i})$ represents the Dirac delta function concentrates at the point $x_{0,i}$.*

(v) *The interior of each domain Γ_i is convex for all $i = 1, \ldots, m$.*

The previous result is within the preprint [4].

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THE EFFECTS OF NON-INTEGRABLE DATA ON THE CRITICAL EXPONENT FOR A σ-EVOLUTION EQUATION WITH STRUCTURAL DAMPING AND NONLINEAR MEMORY

WANDERLEY NUNES DO NASCIMENTO

Universidade Federal do Rio Grande do Sul, Departamento de Matemática Pura e Aplicada, Porto Alegre, RS, Brasil,
wanderley.nascimento@ufrgs.br

Abstract

Our goal in this work is to examine the impact of assuming non-integrable regularity for the initial data on the critical exponent of a σ-evolution equation with structural damping and nonlinear memory.

The key to determining the new critical exponent lies in the interplay between the loss of decay rate, due to the presence of nonlinear memory, and the assumption that the initial data are in L^m rather than L^1. This new critical exponent is different from that found in the L^m theory for the corresponding problem with power nonlinearity $|u|^p$.

We demonstrate the optimality of this critical exponent using the test function method.

1 Introduction

In this work, we are interested in the following $\sigma-$evolution equation with structural damping

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\theta u_t = F(t,u), & t \geq 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\sigma > 0$, $2\theta = \sigma$ and

$$F(t,u) = \int_0^t (t-s)^{-\gamma} |u(s,x)|^p ds, \quad \gamma \in (0,1), \tag{2}$$

represents a memory term, since it is a fractional Riemann-Liouville integral of a power nonlinearity $|u|^p$.

In the case of a power nonlinearity $F = |u|^p$ it has been proved [1] that the critical exponent, assuming small data in L^m, is

$$p_c = 1 + \frac{2m\sigma}{n - 2\sigma}.$$

When we refer to p_c as a critical exponent, we mean that global solutions exist for small initial data when the power is supercritical, i.e., $p > p_c$. In contrast, global solutions do not exist for subcritical powers, $1 < p < p_c$, even with small initial data in the same norm, given appropriate sign assumptions on the data. The critical power $p = p_c$ may either lie within the existence or nonexistence range of global solutions, depending on the specific problem considered.

In [2] the author proved that the critical exponent for (1) in the special case $\theta = \frac{1}{2}$ and $\sigma = 1$, assuming data in L^1, with nonlinear memory term F as in (2), is

$$p_c = \max\{p_\gamma(n), \gamma^{-1}\},$$

where $p_\gamma = 1 + \frac{3-\gamma}{n+\gamma-2}$.

In this talk, we will analyze the influence of the L^m smallness of the initial data. We will conclude that a new critical exponent emerges from the interplay between the loss of decay rate due to the nonlinear memory term and the loss of decay resulting from relaxing the L^1 condition on the initial data.

2 Main Results

If the loss of decay due to the assumption of L^m on the initial data becomes irrelevant with respect to the loss os decay rate related to the presence of the nonlinear memory term, i.e., if

$$\gamma > 1 - \frac{n}{\sigma}\left(1 - \frac{1}{m}\right), \tag{1}$$

the expected critical exponent is

$$p_c = \max\{p^1_{m,\gamma}(n,\sigma), p^2_{m,\gamma}(n,\sigma)\}, \tag{2}$$

where

$$p^1_{m,\gamma}(n,\sigma) \quad = \quad 1 + \frac{\gamma - 3}{\left[1 - \frac{n}{\sigma m}\right]_+} = 1 + \frac{\sigma m(3-\gamma)}{[n - \sigma m]_+} \tag{3}$$

$$p^2_{m,\gamma}(n,\sigma) \quad = \quad 1 + \frac{1-\gamma}{\left[1 - \frac{n}{\sigma}\left(1 - \frac{1}{m}\right)\right]_+} = 1 + \frac{\sigma m(1-\gamma)}{[\sigma m - n(m-1)]_+}. \tag{4}$$

On one hand, we proved that the exponent $p^1_{m,\gamma}(n,\sigma)$ is indeed critical. On the other hand, we only proved the existence of global small data energy solutions when $p > p^2_{m,\gamma}(n,\sigma)$. It remains an open problem to prove that no global weak solutions exist if $1 < p < p^2_{m,\gamma}(n,\sigma)$. The main results are:

Theorem 2.1. *Let $n \le 2\sigma$ and $m \in (1,\infty)$ or $n > 2\sigma$ and $m \in (1, n/(n-\sigma))$. Let us assume (1) and $p > p_c$, or $p > n/(n-2\sigma)$ if $n > 2\sigma$ and $m = \frac{2n}{2\sigma(\gamma-2)+n(3-\gamma)}$. Then there existe $\varepsilon > 0$ such that for any initial data*

$$(u_0,u_1) \in \mathcal{A} = (L^m \cap L^\infty) \cap (L^m \cap L^\infty), \quad \textit{with} \quad \|(u_0,u_1)\|_{\mathcal{A}} := \|u_0\|_{L^m} + \|u_1\|_{L^m} \le \varepsilon,$$

there is a unique global energy solution $u \in \mathcal{C}([0,\infty), H^1) \cap \mathcal{C}^1([0,\infty), L^2)$ to (1). Moreover, it satisfies

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}\left(\frac{1}{m}-\frac{1}{q}\right)}\|u\|_{L^m\cap L^\infty}, \tag{5}$$

for any $q \in [m,\infty]$ if $n < 2\sigma$ and for any $q \in [m, n/(n-2\sigma)]$ if $n \ge 2\sigma$, where $n/(n-2\sigma) = \infty$ if $n = 2\sigma$. If $n \ge 2\sigma$ and $q = \bar{q} = n/(n-2\sigma)$, its satisfies

$$\|u(t,\cdot)\|_{L^{\bar{q}}} \lesssim (1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{m}\right)}\log(e+t)\|u\|_{L^m\cap L^\infty}. \tag{6}$$

If $n > 2\sigma$ and $q \in (n/(n-2\sigma), \infty]$, its satisfies the following decay estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{m}\right)}\|u\|_{L^m\cap L^\infty}. \tag{7}$$

Theorem 2.2. *Let $n \ge 1$, $\gamma \in (0,1)$, $m \in (1,\infty)$. Assume that $(u_0,u_1) \in L^1_{loc}$ with $u_0 \ge 0$, $(u_0(x)+u_1(x)) \ge \varepsilon|x|^{-\frac{n}{m}}\log|x|$, for $|x| >> 1$ and that $u \in L^p([0,\infty)\times\mathbb{R}^n)$ is a global solutioin to (1). Then $p \ge p^1_{m,\gamma}(n,\sigma)$.*

The proofs of these theorems follow the techniques used in [3]

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ON PRIMAL HYBRID FORMULATIONS FOR THE APPROXIMATION OF NEARLY-INCOMPRESSIBLE LINEAR ELASTICITY PROBLEMS

G. TARASCHI[1] & M. R. CORREA[2]

[1]Programa de Doutorado em Matemática Aplicada do IMECC, SP, Brazil, gitaraschi@gmail.com

[2]Departamento de Matemática Aplicada IMECC/Unicamp, SP, Brazil, maicon@ime.unicamp.br

Abstract

In this work, we address some difficulties in approximating nearly-incompressible linear elasticity problems by the Primal Hybrid Finite Element Method. The traditional Primal Hybrid method is generally not robust towards locking, resulting in approximation errors that increase as the first LamÃ© coefficient λ goes to infinity. Here, we explore the viability of two strategies to solve this problem: enriching the displacement approximation space in the traditional hybrid formulation and adding an auxiliary pressure-like field to obtain a hybrid displacement-multiplier-pressure formulation. For the displacement-multiplier-pressure hybrid method in particular, we present a general framework for constructing inf-sup stable spaces, which ensures locking-free approximations.

1 Introduction

Considering $\Omega \subset \mathbb{R}^d$, $d = \{2, 3\}$, a bounded domain with a Lipschitz continuous boundary $\partial\Omega$, the Linear elasticity problem consists of finding the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ such that

$$\operatorname{div}(\boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) = \boldsymbol{f} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{u} = 0 \text{ over } \partial\Omega, \tag{1}$$

where $\boldsymbol{f} \in L^2(\Omega, \mathbb{R}^d)$ represents a distributed load, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the symmetric part of the gradient of $\boldsymbol{u}$, and $\boldsymbol{C}$ is the fourth-order elasticity tensor. For isotropic elastic materials, the elasticity tensor $\boldsymbol{C}$ is described through the Lamé coefficients $\lambda \geq 0$ and $\mu > 0$ according to $\boldsymbol{C\tau} = 2\mu\,\boldsymbol{\tau} + \lambda \operatorname{tr}\boldsymbol{\tau}\,\boldsymbol{I}, \quad \forall \boldsymbol{\tau} \in \mathbb{S},$ where tr denotes the trace operator of a matrix, $\boldsymbol{I}$ is the identity matrix, and $\mathbb{S}$ is the space of second-order symmetric tensors. For nearly-incompressible elastic materials, the first Lamé coefficient λ goes to infinity, resulting in an unbounded elasticity tensor, which may cause loss of accuracy in the numerical approximations for (1), in a phenomenon called volumetric locking. The most classical Finite Element Method to approximate the linear elasticity problem is the Continuous Galerkin method, based on an H^1-conforming variational formulation of problem (1). Alternatively, one may consider methods based on a Primal Hybrid formulation, where the continuity of the displacement is weakly imposed through the addition of Lagrange multipliers [2]. Given $\mathcal{T}_h$ a regular partition of Ω with no hanging nodes, consider the spaces

$$\mathcal{X} = \{\boldsymbol{v} \in L^2(\Omega, \mathbb{R}^d) : \boldsymbol{v}|_K \in H^1(K, \mathbb{R}^d),\ \forall K \in \mathcal{T}_h\} \quad \text{and} \quad \mathcal{M} = \{\boldsymbol{\tau} n^{\partial K}|_{\partial K} : \boldsymbol{\tau} \in H(\operatorname{div}, \Omega, \mathbb{S}) \text{ for all } K \in \mathcal{T}_h\},$$

where $H(\operatorname{div}, \Omega, \mathbb{S})$ is the space of symmetric $H(\operatorname{div})$-conforming second-order tensors and $n^{\partial K}$ is the outwards and unitary normal vector over ∂K. Following the ideas of [2], the primal hybrid formulation for problem (1) consists in finding $(\boldsymbol{u}, \boldsymbol{m}) \in \mathcal{X} \times \mathcal{M}$ such that

$$\sum_{K \in \mathcal{T}_h} \int_K (2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) + \lambda \operatorname{div} \boldsymbol{u} \cdot \operatorname{div} \boldsymbol{v})\,\mathrm{dx} - \sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{m} \cdot \boldsymbol{v}\,\mathrm{ds} = -\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{dx} \qquad \forall \boldsymbol{v} \in \mathcal{X} \tag{2a}$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \boldsymbol{s} \cdot \boldsymbol{u}\,\mathrm{ds} = 0 \qquad \forall \boldsymbol{s} \in \mathcal{M}. \tag{2b}$$

The Primal Hybrid method (PH) is obtained from (2) by replacing $\mathcal{X} \times \mathcal{M}$ with finite-dimensional subspaces $\mathcal{X}_h \times \mathcal{M}_h$, which must satisfy the inf-sup conditions [1, 2] in order to guarantee a unique solution for the associated discrete problem. Unfortunately, even when using inf-sup stable approximation spaces, the PH method generally performs poorly in the incompressibility limit, especially for the approximation of $\boldsymbol{m}$.

An initial strategy to solve this problem is increasing the polynomial degree used in the construction of $\mathcal{X}_h$ while maintaining $\mathcal{M}_h$ unaltered. However, numerical experiments performed in this work showed that such enrichment is not always enough to generate locking-free approximations. This motivates a second strategy, where we modify the variational formulation to introduce an auxiliary pressure-like variable, as originally proposed in [3].

2 Displacement-multiplier-pressure hybrid method

Consider $\mathcal{P} = L^2_0(\Omega, \mathbb{R}^d)$ the space of square-integrable functions with zero average. Introducing the auxiliary variable $p = \lambda \operatorname{div} \boldsymbol{u}$, informally referred to as pressure, it is possible to derive the following three-field formulation

$$\sum_{K\in\mathcal{T}_h} \int_K 2\mu\, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{dx} - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{m}\cdot\boldsymbol{v} \,\mathrm{ds} + \sum_{K\in\mathcal{T}_h} \int_K p \operatorname{div} \boldsymbol{v} \,\mathrm{dx} = -\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} \,\mathrm{dx} \qquad \forall \boldsymbol{v} \in \mathcal{X} \tag{1a}$$

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{s}\cdot\boldsymbol{u} \,\mathrm{ds} = 0 \qquad \forall \boldsymbol{s} \in \mathcal{M} \tag{1b}$$

$$\sum_{K\in\mathcal{T}_h} \int_K q \operatorname{div} \boldsymbol{u} \,\mathrm{dx} - \frac{1}{\lambda} \int_\Omega p\, q \,\mathrm{dx} = 0 \qquad \forall q \in \mathcal{P}. \tag{1c}$$

The displacement-multiplier-pressure hybrid method (DPH) is then obtained by replacing $\mathcal{X} \times \mathcal{M} \times \mathcal{P}$ with finite-dimensional subspaces $\mathcal{X}_h \times \mathcal{M}_h \times \mathcal{P}_h$. Satisfying the inf-sup conditions for the DPH method is central to its convergence, especially for the simulation of nearly-incompressible materials. In the next result, we show a convenient way to obtain stable approximations spaces $\mathcal{X}_h \times \mathcal{M}_h \times \mathcal{P}_h$.

Proposition 2.1. *Let $\mathcal{X}_h \times \mathcal{M}_h$ be an inf-sup stable pair for the original PH method and $\mathcal{U}_h \times \mathcal{P}_h$ an inf-sup stable pair for the Stokes problem, as described in Chapter 8 of [1]. If $\mathcal{U}_h \subset \mathcal{X}_h$, then the composition $\mathcal{X}_h \times \mathcal{M}_h \times \mathcal{P}_h$ is inf-sup stable for formulation (1).*

Using inf-sup stable spaces, it is possible to prove *a priori* error estimates for the DPH method that do not deteriorate as λ goes to infinity. Therefore, the discrete solutions remain accurate even in the nearly-incompressible scenario, resulting in a locking-free method. Such theoretical claims are verified through some simple yet illustrative numerical experiments, from which we conclude that using the displacement-multiplier-pressure formulation is a viable option to deal with problems near the incompressibility limit.

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STATE FEEDBACK AS A STRATEGY FOR COVID-19 CONTROL AND ANALYSIS

L. R. S. RODRIGUES[1] & F. GABRIELLI[1]

[1] Universidade Federal do Maranhão, CCCo, MA, Brazil, leonardo.rodrigues@ufma.br, gabrielli14@gmail.com

Abstract

This paper presents a study on a compartmental epidemic model for COVID-19, examining the stability of its equilibrium points upon vaccination introduction as a strategy to mitigate the spread of the disease. Initially, the Susceptible-Infectious-Quarantine-Recovered (SIQR) mathematical model and its technical aspects are introduced. Subsequently, vaccination is incorporated as a control measure within the model scope. Equilibrium points and the basic reproductive number are determined followed by an analysis of their stability. Furthermore, controllability characteristics and optimal control strategies for the system are investigated, supplemented by numerical simulations.

1 Introduction

Our objective in this work is to consider and analyze the SIQR model properties by adding vaccination as a strategy to control the growth of the disease, study constant solution stability, calculate the basic reproductive number of disease propagation, study system controllability and the conditions to obtain the optimal control and apply the model in some numerical simulations (using MATLAB$^{\text{TM}}$ software) to reach some conclusions about the control method (vaccination).

The model is the following system (1) of ordinary differential equations:

$$\begin{cases} \frac{dS}{dt} = \Delta - \alpha SI - \mu S + v \\ \frac{dI}{dt} = \alpha SI - (\gamma + \mu + \eta) I \\ \frac{dQ}{dt} = (\eta - \epsilon) I - (\rho + \mu) Q \\ \frac{dR}{dt} = \gamma I + \rho Q - \mu R \end{cases} \tag{1}$$

2 Main Results

We considered that $A \in \mathbb{R}^{4,4}$ is the Jacobian matrix of system (1) at E_0 without control perturbation; $B \in \mathbb{R}^{4,2}$ is a real matrix. The function $x : [0,T] \to \mathbb{R}^4$ represents the state, and $u : [0,T] \to \mathbb{R}^2$, the control. Both are vector functions of 4 and 2 components, respectively, depending exclusively on time t.

Let $T > 0$ be a fixed real number, given $t_0 \in [0,T]$ and $x_0 \in \mathbb{R}^4$, considering a dynamic system described by the following differential equations:

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t),\ t \in [0,T] \\ x(t_0) &= x_0, \end{aligned} \tag{1}$$

considering cost functional

$$J(u,x) = \frac{1}{2}\left[\int_0^T x^{\mathbf{T}}(t)G(t)x(t) + u^{\mathbf{T}}(t)R(t)u(t)dt\right] \tag{2}$$

Considering the set of admissible control

$$\mathcal{U}_{ad} := \left\{u \in L^1([0,T];\mathbb{R}^2); u(t) \in \mathcal{X} \subset \mathbb{R}^2 \ a.e \ in \ [t_0,T].\right\}$$

Theorem 2.1. *Let $P(t)$ be a continuous and differentiable symmetric matrix with respect to time t on an interval $[0, T]$, considering the Riccati equation:*

$$\frac{\partial P(t)}{\partial t} = -A^T P(t) - P(t)A + P(t)BR^{-1}B^T P(t) - G$$

in which A is a constant matrix of size $n \times n$; B is a constant matrix of size $n \times m$; R is a positive definite matrix of size $m \times m$; and G is a constant symmetric matrix of size $n \times n$. Optimal control is the form $u^* = -R^{-1}B^{\mathbf{T}}\lambda$. Then, for each initial condition $P(0) = P_0$, there is a unique solution $P(t)$ to the Riccati equation defined by $[0, T]$, associated with control systems (1) to (2).

We see the control is a linear function of the state only, a type of feedback control $u = -R^{-1}B^{\mathbf{T}}Px$. The matrix $R^{-1}B^{\mathbf{T}}P$ is called the *gain*.

In this next simulation, we will perform numerical simulations for an optimal control strategy given by theorem 2.1

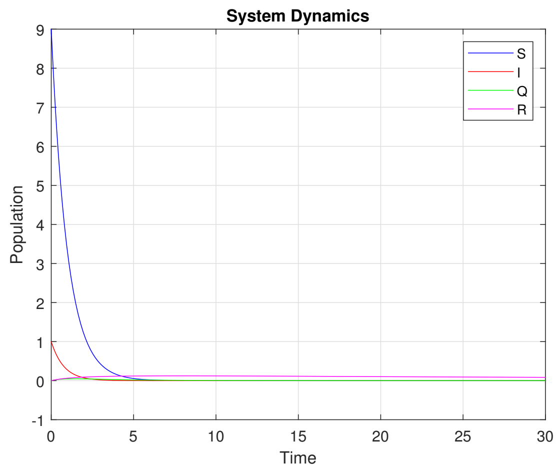

Figure 1: Variational curves of S, I, Q, and R with optimal control

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NUMERICAL SIMULATION OF A BRESSE-TIMOSHENKO SYSTEM WITH THERMOELASTICITY OF TYPE III

R. L. R. MADUREIRA[1], M. A. RINCON[2], H. ZOUGHEIB[3] & T. F. ARWADI[3]

[1]Universidade do Estado do Rio de Janeiro, IME-UERJ, RJ, Brazil, rodrigo.madureira@ime.uerj.br
[2]Universidade Federal do Rio de Janeiro, IC-UFRJ, RJ, Brazil, rincon@ic.ufrj.br
[3]Department of Mathematics, Faculty of Science, Beirut Arab University, Beirut, Lebanon, hmz232@student.bau.edu.lb, t.elarwadi@bau.edu.lb

Abstract

In this work, a numerical simulation is shown for a Bresse-Timoshenko system with thermoelasticity of type III. Some experiments are done to show that this solution is exponentially stable without assuming the condition of equal wave speeds. So, when using a scheme that brings together a finite element approximation in space and finite difference in time, some numerical results are presented to demonstrate the accuracy of the approximation and the behaviour of the solution.

1 Introduction

We consider the Bresse-Timoshenko system with thermodiffusion of type III

$$\begin{cases} \rho_1\phi_{tt} - k(\phi_x+\psi)_x = 0, \\ -\rho_2\phi_{ttx} - b\psi_{xx} + k(\phi_x+\psi) + \beta\theta_{tx} = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \beta\psi_{tx} - \kappa\theta_{txx} = 0; \quad (x,t)\in(0,L)\times(0,\infty), \end{cases} \tag{1}$$

where L represents the distance along the center line of the beam, with Dirichlet boundary conditions

$$\phi(0,t) = \phi(L,t) = \psi(0,t) = \psi(L,t) = \theta(0,t) = \theta(L,t) = 0, \quad t>0, \tag{2}$$

and the initial conditions are

$$\begin{aligned} &\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \quad \psi(x,0) = \psi_0(x), \quad x\in(0,L), \\ &\theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \quad x\in(0,L). \end{aligned} \tag{3}$$

In previous work (see [1]-[5]), researchers did not obtain exponential stability unless they assumed the condition of equal wave speeds, since otherwise we would obtain polynomial stability. In this work, we will show the numerical aspects of the problem (1)-(3) to corroborate that if we take the Bresse-Timoshenko system, that is, Timoshenko system with a free second spectrum, we will obtain exponential stability without any parameter conditions (see [6]).

2 Main Results

2.1 Assumptions

In this work, the existence and uniqueness of the weak solution of the system (1)-(3) were shown. The classical Faedo-Galerkin approximation was used together with a priori estimates and then passing through the limits using compactness arguments. We define $V = H_0^1(0,L)$, $H = L^2(0,L)$ and the Hilbert space $\mathcal{H} := V\times V\times V\times V\times H$. Therefore, the following definition and theorem are necessary.

Definition 2.1. *Let the initial data* $(\phi_0, \phi_1, \psi_0, \theta_0, \theta_1) \in \mathcal{H}$ *then a function* $W = (\phi, \phi_t, \psi, \theta, \theta_t) \in C(0, T; \mathcal{H})$ *is said to be a weak solution of* (1)-(3) *if it is a solution of the weak problem for almost* $t \in [0, T]$.

Theorem 2.1. *Suppose that the initial data colorblue*$(\phi_0, \phi_1, \psi_0, \theta_0, \theta_1) \in \mathcal{H}$ *then system* $(1) - (3)$ *have a weak solution satisfying*

$$\phi \in L^\infty(0, T; V), \ \phi_t \in L^\infty(0, T; V), \ \psi \in L^\infty(0, T; V), \ \theta \in L^2(0, T; V), \ \theta_t \in L^2(0, T; H),$$

where the solution $W = (\phi, \phi_t, \psi, \theta, \theta_t)$ *depends continuously on the initial data in* $\mathcal{H}$. *In particular,* W *is unique solution of system* $(1) - (3)$.

2.2 Numerical simulations

Let the initial conditions given respectively by

$$\begin{aligned} &\varphi(x,0) = \frac{1}{4}\left[\frac{L}{2}\cos\left(\frac{\nu\pi x}{L}\right) + x - \frac{L}{2}\right]; \quad \varphi_t(x,0) = -\frac{1}{4}\left[\frac{L}{2}\cos\left(\frac{\nu\pi x}{L}\right) + x - \frac{L}{2}\right]; \\ &\psi(x,0) = \frac{L}{4\pi}\sin\left(\frac{\nu\pi x}{L}\right) + \frac{x}{2}(x-L)\nu; \quad \theta(x,0) = \frac{1}{4}\sin\left(\frac{\nu\pi x}{L}\right); \quad \theta_t(x,0) = -\frac{1}{4}\sin\left(\frac{\nu\pi x}{L}\right). \end{aligned} \tag{4}$$

The Figure 1 shows the energy decay and its logarithm for $T = 10$ and $h = \Delta t = 2^{-8}$. This experiment corroborates exponential stability without assuming the condition of equal wave speeds (see [6]).

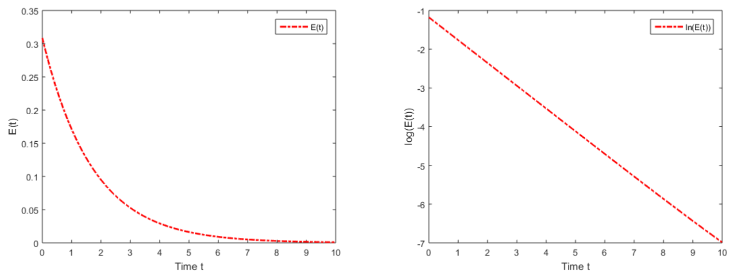

Figure 1: Homogeneous problem - Energy decay and its logarithm

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FINITE ELEMENT SIMULATIONS OF NON-LINEAR VIBRATIONS OF A MEMBRANE

R. S. NEGREIROS[1] & D. G. ALFARO VIGO[2]

[1]Programa de Pós-graduação em Informática, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ, Brasil, rodrigonegreiros23@gmail.com

[2]Instituto de Computação, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ, Brasil, dgalfaro@ic.ufrj.br

Abstract

In this work, we consider the numerical solution of a mixed problem for a non-linear partial differential equation that models the vibrations of a membrane. The numerical scheme uses the finite element method for spatial discretization and the Newmark method for time discretization. For the computational implementation of this numerical scheme, we utilize the FEniCS software. We present numerical examples to highlight some questions that have not been fully addressed in the literature, such as the convergence of the numerical scheme and the decay of solutions.

1 Introduction

We consider the non-linear oscillations of a membrane associated with the bounded region $\Omega \subset \mathbb{R}^2$ and assume that the displacement $u(\mathbf{x}, t)$ for $\mathbf{x} \in \Omega$ and $t \in [0, T]$ satisfies the following mixed problem introduced in [2]:

$$\begin{cases} u_{tt} - \nabla \cdot \sigma(\nabla u) \; - \delta \Delta u_t = f, \quad \text{in } \Omega \times [0,T] \\ u = 0 \text{ on } \Gamma_0 \times (0,T) \quad \text{(Dirichlet boundary condition)} \\ u_{tt} + (\sigma(\nabla u) + \delta \nabla u_t) \cdot \nu = g, \quad \text{on } \Gamma_1 \times (0,T) \quad \text{(Glued-mass boundary condition)} \\ u(x,0) = u^0(x); \quad u_t(x,0) = u^1(x) \quad \text{in } \Omega. \end{cases} \tag{1}$$

where $\sigma(\nabla u) = \left(1 + \epsilon \left|\nabla u\right|^2\right) \nabla u$ is a non-linear constitutive relation and the parameters ϵ, δ control the strength of the non-linearity and internal damping, respectively. The boundary of Ω is decomposed into two disjoint parts Γ_0 and Γ_1, and ν denotes the unitary outward normal to Γ_1. The existence of global solutions to the problem (1) is thoroughly discussed in [2].

2 Numerical scheme

In order to obtain approximate solutions to the problem (1), we apply the finite element method in space combined with the Newmark method for the time discretization, both methods are widely used for the discretization of problems in mechanics [1]. Moreover, at each time step this scheme leads to a system of non-linear equations which is solved by applying the Newton method. For the computational implementation of this numerical scheme, we use the free open source software FEniCS [3] which allows for an efficient automated solution of differential equations.

3 Numerical examples

We perform several tests to validate the computational implementation of the numerical scheme. In Table 1, we summarize some results from the approximation of a synthetic exact solution of problem (1) in the unit square Ω [see Figure 1 (left)] for different discretizations using Lagrange triangular finite elements of degree $G = 1$ with a finite element mesh of size h and a time step of size k. The table shows the L^2 error along with the estimated

h	k	L^2 Error	p
0.1	1.005	0.0149	—
0.05	0.503	0.0037	2.0124
0.025	0.251	0.0009	1.9957
0.0125	0.126	0.0002	1.9957

h	k	L^2 Error	q
0.2	0.04	0.0532	—
0.1	0.02	0.0148	1.8373
0.05	0.01	0.0037	1.9955
0.025	0.005	0.0009	1.9902

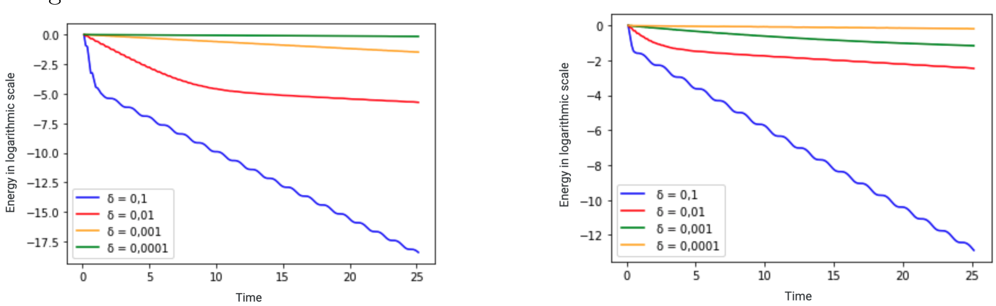
Table 1: Error and estimated order of convergence of the numerical scheme for different discretizations.

convergence orders p and q related to h and k, respectively. We observe that the estimated values are close to $p = G + 1 = 2$ and $q = 2$ which are consistent with known theoretical results for the case $\epsilon = 0$ (linear problem) [3]. To the best of our knowledge, no theoretical convergence result has been published for the non-linear problem.

Figure 1: Examples of finite element meshes for a unit square domain (left) and a unit square domain with a circular hole of radius 0.125 (right). The boundaries for Dirichlet b.c. are represented in red color and for glued-mass b.c. in blue color.

The next set of numerical examples illustrates the decay of the energy of the solutions of problem (1) in the case $f = g = 0$. In Figure 2, we present the results corresponding to different values of δ for the two domains depicted in Figure 1. In these examples, we observe an exponential decay of solutions under conditions less restrictive than those presented in [2] where a truncated version of the function $\sigma(\nabla u)$ was considered.

Figure 2: Time evolution of the logarithm of the scaled energy for the solution of problem (1) for the two domains Ω presented in Figure 1.



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A THEORETICAL LAGRANGIAN-EULERIAN FORMULATION FOR A NON-LOCAL TRAFFIC FLOW MODEL

L. S. ARAUJO[1] & E. ABREU[1]

[1]Universidade Estadual de Campinas, IMECC, PPGMA, Brazil
v233745@dac.unicamp.br, eabreu@unicamp.br

Abstract

In this work, we develop a class of approximations via ODEs on Banach spaces for a non-local differential model:

$$u(x,t)_t + \left[u(x,t)\left(1 - \int_{\mathbb{R}} w(x-\tau)u(\tau,t)d\tau\right)\right]_x = 0 \qquad x \in \mathbb{R}, \quad t > 0, \quad u(0,x) = u_0(x), \tag{1}$$

where $w \in L^1 \cap L^\infty$ is a probability density function. In real-world models, it is natural to assume that the modeling is not fully precise. One way to account for this uncertainty is to introduce a probabilistic term into the underlying equation. The latter approach has been used recently in [4] and [1]. To achieve a robust approximation of model (1) using ODEs, we use the improved concept of No-Flow curves as established recently in literature [5, 6].

1 Introduction

The approximation that we develop under the Lagrangian-Eulerian approach is based on concept of the No-Flow curves [5, 6]. The No-Flow curve is determined by the deduction of a family of curves that define control volumes, which are locally conservative under the function $u(x,t)$, given by the corresponding hyperbolic conservation law $u_t(x,t) + [H(u(x,t))]_x = 0$. On analytic terms, if (σ_i) for $i = 1, 2, \cdots$ is this family of curves, we have

$$\int_{\sigma_i(t)}^{\sigma_{i+1}(t)} u(t,x)dx = \int_{\sigma_i(0)}^{\sigma_{i+1}(0)} u(0,x)dx, \qquad t \in (0,\infty).$$

If we apply the derivative on the equality above and impose orthogonality of the curves with the flow we can achieve the family of ODEs

$$\frac{H(u(\sigma(t),t))}{u(\sigma(t),t)} = \sigma'(t), \qquad \sigma(0) = x_0. \tag{2}$$

For the use of the No-Flow curves on an approximation of PDEs such as (1), we use a dimensional analysis of the the quantity $H(u)/u$ and we have $\Delta x/\Delta t \propto \mathcal{O}(H(u)/u)$. This allow us to reach semi-discrete schemes (No-Flow ODEs on Banach spaces) from fully-discrete schemes, by replacing the terms $\Delta x/\Delta t$ for $H(u_k)/u_k$ and taking the rigorous limit on $\Delta t \to 0$ [5, 3, 6, 2].

2 Main Results

For our particular purpose, we adapt the argument made on the local case [5, 6] for the non-local traffic flow model (1). Define $N = 1/\varepsilon$, $G(x) = 1 - x$, and we use the following discretization of the convolution integral,

$$1 - \int_{\mathbb{R}} w(x-\tau)u(\tau,t)d\tau \simeq V_N(u)(x,t) = \sum_{k=0}^{N} G(u(x+k\varepsilon,t))W_k, \qquad W_k = \int_{k\varepsilon}^{(k+1)\varepsilon} w_n(t)dt,$$

Our Lagrangian-Eulerian approach give us the following family of ODEs on the conservative form

$$(u_\varepsilon)_t = -\frac{g_N(u_\varepsilon, u_{\varepsilon+1}) - g_{N-1}(u_{\varepsilon-1}, u_\varepsilon)}{\varepsilon}, \qquad u(x,0) = u_0(x), \tag{3}$$

with $u_{\varepsilon+k} = u_{\varepsilon+k}(x,t) = u_\varepsilon(x+k\varepsilon)$ and g_N defined by

$$g_N(x,y) = \frac{xV_N(u_{\varepsilon+1}) - yV_N(u_\varepsilon)}{4} + \frac{1}{4}\left(V_N(u_\varepsilon) + V_N(u_{\varepsilon+1})\right)(x+y). \tag{4}$$

For to ensure convergence we use the following CFL condition:

$$\frac{\Delta t}{\varepsilon}(1/2 + M + M^2) \leq \frac{1}{3}, \tag{5}$$

where $M = ||u_0||_\infty$. A summarize the results about the approximation (3) on the following theorem:

Theorem 2.1. *Under the hypothesis of $u_0(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ and the CFL condition (5) the family of solutions of (3) converges as $\varepsilon \to 0$ for the unique entropy solution of (1) and attain to:*

1) $||u_\varepsilon(\cdot,t)||_{L^1(\mathbb{R})} \leq ||u_0||_{L^1(\mathbb{R})}$, *2) total variation bound $TV(u_\varepsilon(\cdot,t)) \leq TV(u_0)$,*

3) Recover the non-local entropy

$$\int_{\mathbb{R}}\int_0^T |u-c|\phi_t dtdx + |u-c|V(u)\phi_x - sign(u-c)[V(u)]_x c\phi(x)dxdt \geq 0, \quad c \in \mathbb{R}.$$

The proof can be found in [3, 2].

3 Conclusions

We achieve a convergence and stability results for an approximation of a non-local traffic-flow model under the same approach as made for local models. We obtained a new analytical-numerical approach for solving convolution non-local models of type (1) discussed on [4, 1]. For a future work, we are studying the No-Flow approach as a novel tool for solving Hyperbolic conservation laws with discontinuous flux functions and a BV criterion for ensure uniqueness of solution of problem (2). This project was funded in part (E.A. by a CNPq grant 307641/2023-6 and L.A by an institutional CAPES MSc fellowship).

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FREE BOUNDARY AND RATTLING PATTERNS IN PARABOLIC EQUATIONS WITH HYSTERESIS

S. TIKHOMIROV

Pontificia Unidersidade Catolico do Rio de Janeiro - Rio de Janeiro, RJ - Brasil, sergey.tikhomirov@gmail.com

Abstract

We consider reaction-diffusion equations with hysteresis nonlinearity. Hysteresis naturally appears as a mechanism of self-organization and is often used in control theory. Important features of the hysteresis operator are discontinuity and memory. Due to the discontinuity of hysteresis, questions of well-posedness of such equations are highly non trivial. For so-called transverse initial data we establish the existence and uniqueness of the solution. Important part of the proof is the free boundary problem. For non-transverse initial data we consider a spatial discretization of the problem and present a new mechanism of pattern formation, called "rattling". The profile of hysteresis forms a highly oscillating quasiperiodic pattern. "Rattling" is very robust and persists in arbitrary dimensions and in systems acting on different time scales. The presentation is based on the joint work with P. Gurevich.

1 Introduction

Hysteresis $\mathcal{H}(u)(t)$ for a real-valued function $u(t)$ is defined as follows, Fig. 1.a. One fixes two thresholds $\alpha < \beta$ and two outputs $h_1 \neq h_{-1}$. If $u(t) \le \alpha$, then $\mathcal{H}(u)(t) = h_1$; if $u(t) \ge \beta$, then $\mathcal{H}(u)(t) = h_{-1}$; if $u(t)$ is between α and β, then $\mathcal{H}(u)(t)$ takes the same value as "just before." The main features of systems with hysteresis are the dependence of the output on the prehistory of the input, rate independence, and nonsmoothness. Hysteresis operators are applied in mathematical descriptions of various physical, chemical and biological processes: thermocontrol, chemical reactors, ferro-magnetism, self-organisation and others.

Parabolic equations with hysteresis appear in models with several diffusive and nondiffusive substances that interact according to a hysteresis law. On a bounded domain $\Omega \subset \mathbb{R}^n$, consider the prototype model

$$u_t = D\Delta u + f(u,v), \quad v = \mathcal{H}(u), \tag{1}$$

supplemented with the initial and boundary conditions, where $D > 0$ and f is a smooth nonlinearity. It is important that the input $u(x,t)$ of the hysteresis $\mathcal{H}$ is not only a function of time, but also of space. If we regard x as a parameter, we can define $\mathcal{H}(u(x,\cdot))(t)$ as above. A system of type (1) was formulated for the first time in [1], in order to model the growth of a colony of bacteria in a Petri plate. Rigorous analysis of (1) appears to be very nontrivial because the hysteresis may switch at different spatial points at different time moments. Due to the discontinuous nature of the hysteresis, the well-posedness of system (1) is a nontrivial question. To overcome those difficulties, one usually replaces hysteresis by a properly defined multi-valued map. This allows one to establish existence of a solution [2]; uniqueness and continuous dependence of the solution on initial data remains open.

2 Transverse initial data

In [5], we observed that the dynamics of solutions of system (1) is related to the evolution of the free boundary that separates subdomains in Ω, in each of which the hysteresis is a constant. This observation suggests a connection with parabolic obstacle-type problems [3]. For reaction-diffusion equations with hysteresis, the free-boundary approach allowed us to distinguish a wide class of so-called transverse initial data (those initial data that cross the thresholds α and β with nonzero slope) for which (1) is well posed in one spatial dimension Ω [4, 6].

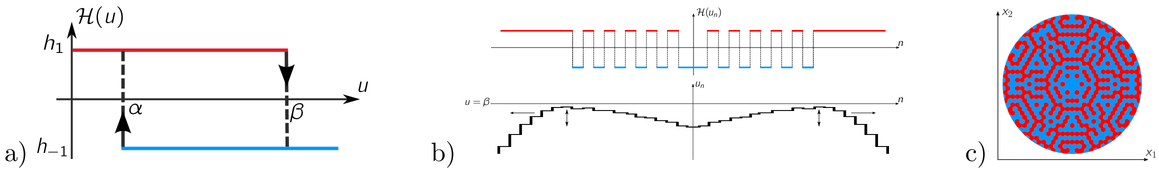

Figure 1: a) Hysteresis $\mathcal{H}$. b) Rattling for $h_{-1} = -h_1$: a) profiles of hysteresis and solution, c) hysteresis pattern for $h_{-1} = -h_1$.

3 Rattling

The nontransverse case is a completely open problem even in the one-dimensional case. Preliminary numerical results indicate that, in order to guarantee well-posedness, the operator $\mathcal{H}$ must be redefined in the nontransverse case. To find the proper definition of $\mathcal{H}$, we begin with the spatially discretized system. For clarity of the explanation, take u to be a scalar function, $D = 1$ and $f(u, v) = v$. Set $\beta = 0$ and consider the nontransverse initial data $u|_{t=0} = -cx^2$ for $x \in \mathbb{R}$ and some $c > 0$. Setting $u_j(t) = u(\delta j, t)$ for a fixed $\delta > 0$, we rewrite (1) as

$$\begin{cases} \dot{u}_j = \dfrac{u_{j-1} - 2u_j + u_{j+1}}{\delta^2} + \mathcal{H}(u_j), \\ u_j(0) = -c(\delta j)^2, \quad j \in \mathbb{Z}, \end{cases} \tag{2}$$

which is a lattice dynamical system. Numerical simulations of system (2) exhibit a remarkable behavior (Fig. 1.b):

$$\frac{\text{number of nodes that do not switch}}{\text{number of nodes that switch}} \approx \left| \frac{h_{-1}}{h_1} \right|. \tag{3}$$

This is a very rigid phenomenon, which persists in multidimensional domains and seems to be independent of the shape of the lattice, Fig. 1.c. Rattling is a new mechanism of pattern formation, its nature is different from other mechanisms, such as travelling waves, Turing instability, etc. We proved the conjecture (3) for the case $h_{-1} = 0$ [7]. In this case all the nodes switch. The rattling effect in (2) gets finer and finer as $\delta \to 0$. Therefore, in the limit of the case $h_{-1} < 0$ there is no hope to obtain the function $\mathcal{H}(u)$ taking values h_1 or h_{-1} on a measurable set, but $\mathcal{H}(u)$ should take an intermediate value (zero in the case of a scalar equation).

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MEASURE NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

F. G. DE ANDRADE[1], M. FEDERSON[2], M. V. S. FRASSON[2], ROGELIO G. ACUÑA[3] & P. H. TACURI[4]

[1]Colégio Técnico de Bom Jesus, Universidade Federal do Piauí, Bom Jesus, PI, Brazil, andrade.fg89@gmail.com
[2]Universidade de São Paulo, ICMC, SP, Brazil, federson@icmc.usp.br, frasson@icmc.usp.br
[3]Universidad del Norte, DME, Barranquilla, Colombia, rogeliograu@gmail.com
[4]Universidade Estadual de Maringá PR, Brazil, phtcordova2@uem.br

Abstract

Measure differential equations is a recently discovered branch of differential equations that can be used as a tool to study physical models more closely aligned with reality, such as models exhibiting jump phenomena. Although this field has developed in recent years, the theory of measure functional differential equations remains scarce, and some classes of these equations have yet to be described. In this work, we explore neutral measure functional differential equations with infinite delay. Using techniques known in the literature, we obtain qualitative properties of their solutions, such as existence, uniqueness, and continuous dependence.

1 Introduction

Every study of Ordinary Differential Equations (ODE) has come across some phase portrait. It is common to find figures on the covers of books on this topic that represent trajectories in phase space. The history of the term "phase space" is unclear and is intertwined with the development of multidimensional spaces, involving great names such as Joseph Liouville, Pierre de Fermat, Ludwig Boltzmann, James Clerk Maxwell, Henri Poincaré and Paul Ehrenfest [2]. Phase space is an important tool in the study of dynamic systems (which can be biological, physical, economic models, etc.) when the intention is to understand qualitative behavior and obtain information about the properties of trajectories, such as stability and bifurcations. Roughly speaking, phase space is a multidimensional space in which each point represents a complete state of a physical system.

When we talk about functional differential equations (FDE) with finite delay time, instead of a point in a multidimensional (finite-dimensional) space, the state is represented by a function in an infinite-dimensional space. For example, if $\tau > 0$ is the delay time of an FDE, then the phase space could be the space $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ of continuous functions defined in $[-\tau, 0]$ that take values in $\mathbb{R}^n$. In this case, the state at time r is determined by the function x_t defined in $[t-\tau, t]$. Thas is, phase space describes the state of the system at a past time. We can think of pahse space as a generalization of the concept of ODE, considering $\tau = 0$.

In the case of EDF's with infinite delay, still within the space of continuous functions, [4] and [2] have axiomatically defined phase spaces. These spaces were used by many authors later, such as [3].

Expanding these ideas and moving on to differential equations in measure with infinite delay, whose solutions are no longer continuous functions, [5] proposes a phase space. In this work, we will use the phase space, also defined axiomatically, proposed by [1].

2 Main Results

A measure neutral functional differential equation with infinite delay, here abbreviated by NFDE, is an equation of the form

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y_s)\, dg(s) + N(t)y_t - N(t_0)y_{t_0}, \tag{1}$$

where $g\colon [t_0, t_0+\sigma] \to \mathbb{R}$ is a non decreasing function, $f, N\colon [t_0, t_0+\sigma]\times\mathcal{B} \to \mathbb{R}^n$ are continuous applications, with $N(t)\colon \mathcal{B}\to\mathbb{R}^n$ bounded linear for each $t\in[t_0, t_0+\sigma]$. If $y_{t_0} = \varphi \in \mathcal{B}$, we say (1) has de the initial condition φ.

Let $\mathcal{O}\subset H$ a nonempty bounded subset, where

$$H = H(t_0,\sigma) = \left\{y\colon (-\infty, t_0+\sigma)\to\mathbb{R}^n\colon y|_{[t_0,t_0+\sigma]} \text{ is regulated and } y_{t_0}\in\mathcal{B}\right\}.$$

Our goal is to associate with the equation (1) a generalized ODE

$$\frac{dx}{d\tau} = DF(t,x), \quad t\in[t_0, t_0+\sigma], \tag{2}$$

where $x\colon [t_0, t_0+\sigma]\to\mathcal{O}$ and $F\colon [t_0,t_0+\sigma]\times\mathcal{O}\to G((-\infty, t_0+\sigma], \mathbb{R}^n)$ is given by

$$F(t,x)(v) = \begin{cases} 0, & v\in(-\infty, t_0),\\ \displaystyle\int_{t_0}^{v} f(s,x_s)\,dg(s) + N(v)x_v, & v\in[t_0,t],\\ \displaystyle\int_{t_0}^{t} f(s,x_s)\,dg(s) + N(t)x_t, & v\in[t, t_0+\sigma], \end{cases} \tag{3}$$

according with the following relation, between the solution x of (2) and the solution y of (1),

$$x(t)(v) = \begin{cases} y(v), & v\in(-\infty, t],\\ y(t), & v\in[t, t_0+\sigma]. \end{cases}$$

The function F defined in (3) is well-defined so that it is possible establish that the sentences composing the function F really take values in the space of regulated functions.

Also, we will restrict ourselves to the study of equations whose F function in (2) satisfies two common conditions in generalized ODE theory and in order to ensure a correspondence between (1) and (2) will assume the some conditions over the bounded of the Kurzweil integrals which appear in (3).

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EXISTENCE AND REGULARITY OF SOLUTION FOR A SEMILINEAR ELLIPTIC EQUATION WITH SINGULAR NONLINEARITY

L. H. MIRANDA[1] & J. T. S. OLIVEIRA[1]

[1]Universidade de Brasília, MAT, DF, Brasil, l.h.miranda@mat.unb.br, jadde@aluno.unb.br

Abstract

In this work, we investigate the existence and regularity of solutions for a semilinear elliptic equation with singular nonlinearity, following the studies of Lucio Boccardo and Luigi Orsina in [2]. This problem is given by:

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = \frac{f(x)}{u^\gamma}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded subset of $\mathbb{R}^N$ of class C^1, $N \geq 2$, $f : \Omega \longrightarrow \mathbb{R}$ is a function belonging to some Lebesgue Space, $\gamma > 0$ and M is a bounded elliptic matrix.

1 Introduction

In this work, following the studies of Lucio Boccardo and Luigi Orsina in [2], we will investigate the existence and regularity of the solution of the following semilinear elliptical problem with singular nonlinearity

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) = \frac{f(x)}{u^\gamma}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open of class C^1, $N \geq 2$, $\gamma > 0$, $f \in L^m(\Omega)$, $m \geq 1$, and M is a bounded elliptic matrix, that is, there are $\alpha, \beta > 0$ such that

$$\alpha|\xi|^2 \leq M(x)\xi \cdot \xi, \quad |M(x)| \leq \beta \quad \forall\, \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega.$$

In the same way as [2], the Problem (1) was divided into three parts with respect to γ, when it is equal to 1, greater than 1 and less than 1. To obtain the results, we initially consider the following approximate problem

$$\begin{cases} -\,\mathrm{div}(M(x)\nabla u_n) = \frac{f_n}{(u_n + \frac{1}{n})^\gamma} & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

where f is a non-negative measurable function, $n \in \mathbb{N}$, $f_n = \min\{f, n\}$ and M is a bounded elliptic matrix.

2 Main Results

Below, we highlight the main results discussed in this work.

Proposition 2.1. *Problem (2) has a unique solution $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ non-negative, so that*

i) *u_n is increasing with respect to n;*

ii) *$u_n > 0$ in Ω;*

iii) *For every $\widehat{\Omega} \subset\subset \Omega$, there is a $K_{\widehat{\Omega}} > 0$, independent of n, such that*

$$u_n(x) \geq K_{\widehat{\Omega}} > 0 \tag{1}$$

for all $x \in \widehat{\Omega}$ and for all $n \in \mathbb{N}$.

Proof The key to demonstrating the existence of $K_{\widehat{\Omega}}$ is a Maximum Principle in [3]. ■

Theorem 2.1. *In Problem (1), consider $f \in L^1(\Omega)$ a non-negative function not identically null and $\gamma = 1$. Then there is a solution $u \in W_0^{1,2}(\Omega)$ in the sense of*

$$\int_\Omega M(x)\nabla u \cdot \nabla \phi = \int_\Omega \frac{f\phi}{u} \quad \forall\, \phi \in C_c^\infty(\Omega).$$

Proof Proving that u_n is limited in $W_0^{1,2}(\Omega)$, it is sufficient to use the weak convergence on the left side of the above equality and use Lebesgue's Dominated Convergence Theorem. The conclusion is immediate by Proposition 2.1. ■

Theorem 2.2. *Let $\gamma > 1$ and $f \in L^1(\Omega)$ be non-negative and not identically null. Then there is a solution $u \in W_{loc}^{1,2}(\Omega)$ of Problem (1) in the sense of*

$$\int_\Omega M(x)\nabla u \cdot \nabla \phi = \int_\Omega \frac{f\phi}{u} \quad \forall\, \phi \in C_c^\infty(\Omega).$$

Furthermore, $u = 0$ in $\partial\Omega$.

Proof Proving that u_n is bounded in $W_{loc}^{1,2}(\Omega)$, $u_n^{\frac{\gamma+1}{2}}$ is bounded in $W_0^{1,2}(\Omega)$ and using the definition of $u = 0$ in $\partial\Omega$, the result follows. ■

Theorem 2.3. *Considering the Problem (1) with $\gamma < 1$ and $f \in L^m(\Omega)$ non-negative and non-identically null, where $m = 2N/[N + 2 + \gamma(N-2)] = [2^*/(1-\gamma)]'$, there is a solution $u \in W_0^{1,2}(\Omega)$ in the sense of*

$$\int_\Omega M(x)\nabla u \cdot \nabla \phi = \int_\Omega \frac{f\phi}{u} \quad \forall\, \phi \in C_c^\infty(\Omega).$$

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A MATHEMATICAL MODEL OF HIV/AIDS SPREAD IN CHILD AND ADULT POPULATION

J. F. PACAZUCA[1] & C. C. ESPITIA[2]

[1]Laboratorio Nacional de Computação Científica, Petrópolis, Brazil, juanpaca@posgrad.lncc.br
[2]Universidad de los Llanos, Villavicencio, Colombia, espitiacristian@gmail.com

Abstract

In this paper we present a mathematical model to study the dynamics of Human Immunodeficiency Virus (HIV) and Acquired Immune Deficiency Syndrome (AIDS). This model incorporates vertical transmission along with nonlinear dynamics governing transmission between adults, individuals in treatment and those living with the disease. Employing rigorous mathematical analysis techniques, we explore crucial epidemiological metrics such as basic reproduction number, equilibrium points, and conditions that dictate infection persistence or extinction, which are inferred through stability analysis. Simulations and sensitivity analysis results shed light on the nuanced interplay of these factors, providing insight into the intricate dynamics of transmission. In particular, our findings underscore the great importance of adult-to-adult transmission in influencing disease dynamics, overshadowing other contributing factors. Furthermore, we found that birth rate and mother-to-child transmission exert a comparatively less significant impact on the escalation of infections.

1 Introduction

Transmission of HIV/AIDS can occur in several ways, including sexual transmission, parenteral transmission, where the primary way it occurs is needle sharing in injection drug users, and finally, vertical transmission from mother to child during pregnancy, childbirth or breastfeeding. [1] state that although sexual transmission is the most common route of HIV infection, needle-sharing and vertical transmission also contribute significantly to the spread of the virus. For example, [2], [3] claim that in some cities, the prevalence of HIV/AIDS among people who inject drugs is significantly higher than in the general population, indicating the importance of parenteral transmission in the HIV epidemic.

2 Main Result

2.1 A Model with Child and Adult Population

Two age groups are considered: 1. Adults (over 13 years of age), who are infected by sexual intercourse and by sharing infected needles in injecting drug users. 2. Children who are infected only by vertical transmission.

Regarding the notation of variables, the total population is $N(t)$, and is divided into 6 compartments, S means susceptible, I means infected, T are treated with antirretroviral therapy and finally A are individuals living with AIDS, the subscript c means children.

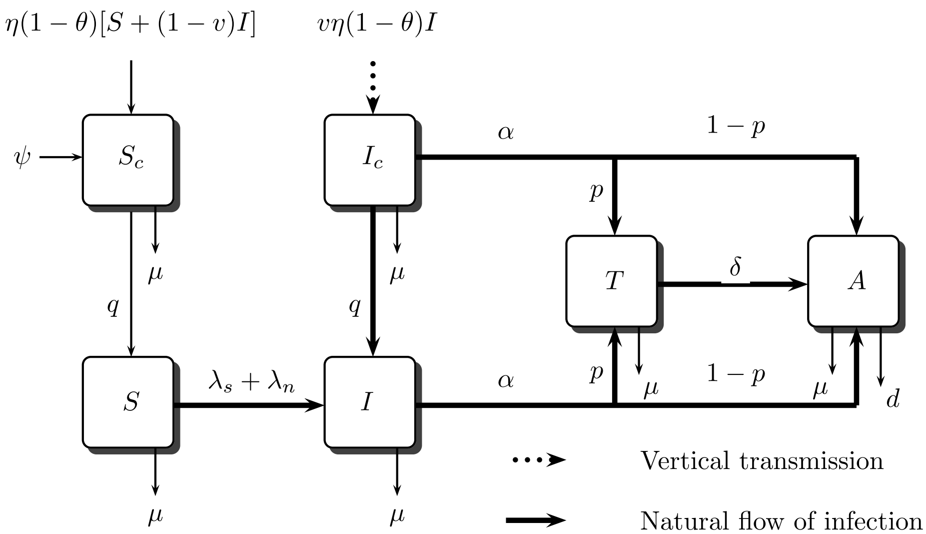
Infected partners who are on antirretroviral treatment are less likely to be infected than those who are not on treatment. Thus, is assumed that treated and untreated individuals living with AIDS are outside of the transmission dynamics. Therefore, the forces of infection, it is the form how the infection is transmitted and mathematically modeled, will be as follows:

$$\lambda_s = \frac{c_s b_s I}{S+I} \quad \text{(Sexual Contagion)} \quad \lambda_n = \frac{c_n b_n I}{S+I} \quad \text{(Contagion by needle sharing in IDU)} \tag{1}$$

To summarize, we consider that $\lambda_s + \lambda_n = \frac{\beta I}{S+I}$ where $\beta = c_s b_s + c_n b_n$. (2)

The above information allows for the construction of the following compartmental diagram, figure 1, which is proposed that adequately explains the dynamics of the model.

Figure 1: Flux diagram.



With initial conditions given by $\Big(S_c(0), I_c(0), S(0), I(0), T(0), A(0)\Big) \in \mathbb{R}^6_+$ that give us some numerical results:

Figure 2: Varying the β parameter in untreated infected children.

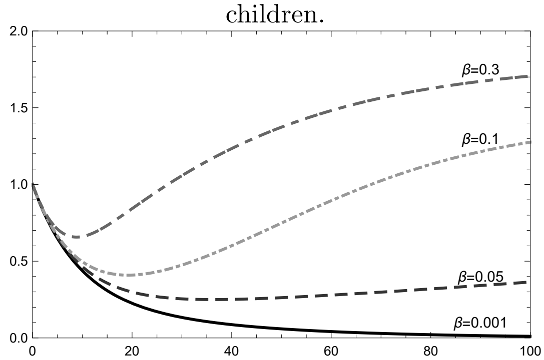
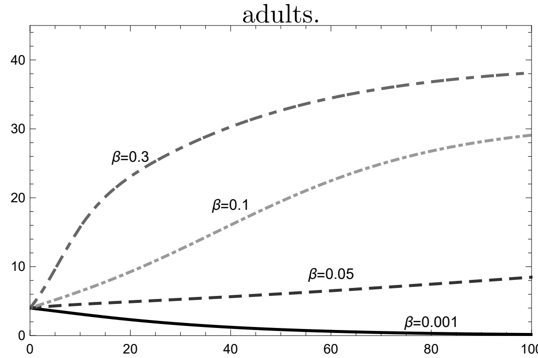


Figure 3: Varying the β parameter in untreated infected adults.



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SHARP POINT-WISE BEHAVIOR OF THE POSITIVE SOLUTIONS OF A CLASS OF DEGENERATE NON-LOCAL ELLIPTIC BVP'S

W. CINTRA[1], J. LÓPEZ-GÓMEZ[2], C. A. SANTOS[1] & L. SANTOS[3]

[1]Universidade de Brasília, DF, Brazil, willian@unb.br, csantos@unb.br
[2]Universidad Complutense de Madrid, Madrid, Spain, julian@mat.ucm.es
[3]Universidade Federal de Viçosa, MG, Brazil, matmslais@gmail.com

Abstract

This work investigates a degenerate non-local boundary value problem of logistic type. The presence of the non-local term prevents us from using the classical sub and supersolution methods to characterize the existence of positive solutions and ascertain their point-wise behavior with respect to on parameters of the problem. Combining some ideas from [2] with the theory of large solutions from [3], we conduct a detailed study of the asymptotic point-wise behavior of positive solutions, revealing a behavior substantially different from that exhibited by the positive solutions of the underlying local problem due to the presence of the non-local term.

1 Introduction

In this work we analyze the existence, the uniqueness, and the limiting point-wise behavior in λ of the positive solutions of the non-local degenerate boundary value problem

$$\begin{cases} -\Delta u = \lambda u - b(x)u^p - a(x)u \displaystyle\int_\Omega c(y)|u(y)|^r \, dy & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary, $p > 1$, $r \geq 1$, a, $b \in \mathcal{C}^\nu(\overline{\Omega})$, $\nu \in (0, 1]$, vanish in some subsets of positive measure of Ω, say $\overline{\Omega}_{0,a}$ and $\overline{\Omega}_{0,b}$, and $0 < c \in L^\infty(\Omega)$.

The problem (1) provides us with the steady-states of a reaction-diffusion model in Population Dynamics. In these models, Ω models the inhabiting territory of a species whose density at the position $x \in \Omega$ is given by $u(x)$. As homogeneous Dirichlet boundary conditions are considered on $\partial\Omega$, the surroundings are assumed to be hostile for the species u. The diffusion term on the left side hand of (1) describes a random spatial movement of the individuals of u, while the *reaction term* on the right hand side provides us with the local reproduction rate per individual, i.e., the per capita population growth rate. As observed in [1], the "crowding effect" of the population is given by $g(x, u, \int_\Omega c(x)u^r) = 1 - (a(x)\int_\Omega c(x)u^r + b(x)u^{p-1})/\lambda$. It turns out that this is the most direct way of introducing the non-local effects. Adopting this perspective, the term

$$C(x, , u, \int_\Omega c(x)u^r) = \frac{\lambda}{b(x)u^{p-1} + a(x) \displaystyle\int_\Omega c(x)u^r}$$

measures the carrying capacity at each $x \in \Omega$, which is related to the amount of individuals that the region can support so that the higher the value of C, the more individuals can inhabit that zone. As we are assuming that both a and b may be simultaneously zero in certain patches, we are allowing C to be infinite on some regions of the domain, referred to as "refuge zones" or "protected areas", where the species grows according to the Malthus law. Naturally, where C is finite, the species grows according to the Verhulst law. We emphasize that, in the model (1), the carrying capacity depends on a sort of weighted nonlinear average of the total population, $\int_\Omega c(y)u^r(y)\, dy$.

2 Main Results

To state the main findings of this work, we need to introduce the following notation. Define $\Omega_0 := \Omega_{0,a} \cap \Omega_{0,b}$, which might be empty, and consider

$$\sigma_1 := \sigma_1[-\Delta] \quad \text{and} \quad \sigma_1^0 := \sigma_1[-\Delta; \Omega_0],$$

where $\sigma_1^0 = \infty$, if $\Omega_0 = \emptyset$. With respecto to the existence of positive solution, we have:

Theorem 2.1. *Under the above assumptions, the problem (1) possesses a positive solution in $C^{2,\nu}(\overline{\Omega})$ if, and only if, $\lambda \in (\sigma_1, \sigma_1^0)$. Moreover, it is unique if it exists, and if we denote it by u_λ, then*

$$\mathfrak{C} := \{(\lambda, u_\lambda) \ : \ \lambda \in (\sigma_1, \sigma_1^0)\}$$

is a continuous curve in $\mathbb{R} \times C^{2,\nu}(\overline{\Omega})$ such that

$$\lim_{\lambda \downarrow \sigma_1} \|u_\lambda\|_\infty = 0, \qquad \lim_{\lambda \uparrow \sigma_1^0} \|u_\lambda\|_\infty = \infty. \tag{2}$$

Furthermore, if:

(a) $\Omega_0 \neq \emptyset$, *then*

$$\lim_{\lambda \uparrow \sigma_1^0} \frac{u_\lambda}{\|u_\lambda\|_2} = \begin{cases} \varphi_0 & \text{in } \bar{\Omega}_0, \\ 0 & \text{in } \Omega \setminus \Omega_0, \end{cases} \tag{3}$$

where φ_0 stands for the principal eigenfunction associated to σ_1^0 normalized so that $\int_{\Omega_0} \varphi_0^2 = 1$.

(b) $\Omega_0 = \emptyset$, *then*

$$\lim_{\lambda \uparrow \infty} u_\lambda = +\infty \quad \text{uniformly in compact subsets of } \Omega_{0,a}.$$

Now, regarding the asymptotic behavior of the solutions, we have the following result:

Theorem 2.2. *Assume that $\Omega_0 \neq \emptyset$ is a smooth subdomain of Ω. Then:*

(a) For every smooth subdomain D of $\mathbb{R}^N$ such that $\overline{D} \subset \Omega \setminus \overline{\Omega}_{0,b}$, there exists a constant $C(D) > 0$ such that

$$\sup_{\lambda \in (\sigma_1, \sigma_1^0)} \|u_\lambda\|_{L^\infty(D)} \leq C(D).$$

(b) For every compact subset $K \subset \Omega \setminus (\overline{\Omega}_{0,a} \cup \overline{\Omega}_{0,b})$, $\lim_{\lambda \uparrow \sigma_1^0} \|u_\lambda\|_{L^\infty(K)} = 0$.

(c) If $r > 1$ and $c^{\frac{1}{1-r}} \in L^1(\Omega)$, then, for every compact subset K of $\Omega \setminus \overline{\Omega}_{0,a}$, $\lim_{\lambda \uparrow \sigma_1^0} \|u_\lambda\|_{L^1(K)} = 0$.

This theorem leads us to the following interpretation from the point of view of population dynamics in a model with refugee zone (i.e., $\Omega_0 \neq 0$). It can be proved that the average of the total population, measured by $\int_\Omega c u_\lambda^r$, increases monotonically to infinity as $\lambda \uparrow \sigma_1^0$. Such growth leads to a devastating effect of the population in the region $a > 0$, that becomes inhospitable for the species as $\lambda \uparrow \sigma_1^0$ (Theorem 2.2 (a) and (b)).

We also obtain a complete description of the asymptotic behavior in the particular case $\overline{\Omega}_{0,a} \subset \Omega_{0,b}$.

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ON THE LINEAR GROWTH OF THE MIXING ZONE IN A SEMI-DISCRETE MODEL OF INCOMPRESSIBLE POROUS MEDIUM (IPM) EQUATION

Y. PETROVA[1], S. TIKHOMIROV[1] & YA. EFENDIEV[2]

[1]PUC-Rio, Department of Mathematics, RJ, Brazil, yu.pe.petrova@gmail.com, sergey.tikhomirov@gmail.com
[2] Texas A&M University, Department of Mathematics & ISC, USA, efendiev@math.tamu.edu

Abstract

The presentation is devoted to viscous / gravitational fingering phenomenon - the unstable displacement of miscible liquids in porous media with the speed determined by Darcy's law. Laboratory and numerical experiments show the linear growth of the mixing zone, and we are interested in determining the exact speed of propagation of fingers. One of the possible mechanisms of slowing down the fingers' growth is due to convection in the transversal direction, that we try to explain by introducing a semi-discrete model of incompressible porous medium equation (IPM). In the simplest setting we show the structure of gravitational fingers - the mixing zone consists of space-time regions of constant intermediate concentration and the profile of propagation is characterized by two consecutive travelling waves which we call a terrace. Based on joint work with S. Tikhomirov and Ya. Efendiev (arXiv:2401.05981).

1 Introduction

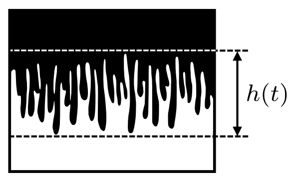
We present a semi-discrete model (1)-(4) of the two-dimensional viscous incompressible porous medium (IPM) equation describing gravitational fingering instability. The IPM equation describes evolution of concentration carried by the flow of incompressible fluid determined via Darcy's law in the field of gravity:

$$\partial_t c + \operatorname{div}(uc) = \nu \Delta c, \quad \operatorname{div}(u) = 0, \quad u = -\nabla p - (0, c). \tag{1}$$

Here $c = c(t, x, y)$ is the transported concentration, $u = u(t, x, y)$ is the vector field describing the fluid motion, $p = p(t, x, y)$ is the pressure, and $\nu \geq 0$ is an inverse of the Peclet number. Usually the spatial domain (x, y) is either the whole space $\mathbb{R}^2$ or cylinder $[0, 1] \times \mathbb{R}$ with periodic conditions, but here we consider a discretization in x.

We are interested in studying the *exact rate of the linear growth of mixing zone* formed when the initial condition is close to the unstable stratification:

$$c(0, x, y) = \begin{cases} +1, & y \geq 0, \qquad \text{(heavy fluid)} \\ -1, & y < 0. \qquad \text{(light fluid)} \end{cases} \tag{2}$$



A theoretical approach to estimate the width $h(t)$ of the mixing zone was done in [2]. In particular, the authors get $h(t) \leq 4t$ using energy estimates (no proven pointwise estimates). Quantification of the size of the mixing zone in laboratory and numerical experiments for 2D case does not give exact answer: it shows that (see e.g. [3])

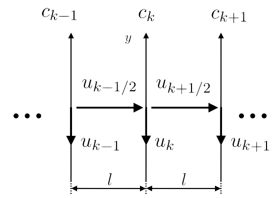
$$h(t) \sim \alpha t, \quad \text{for some } \alpha \in [1.34, 2]. \tag{3}$$

The semi-discrete model of IPM that we introduce in Sec. 2 explains the possible mechanism of slowdown of fingers and has potential to give better estimates for the size of the mixing zone $h(t)$.

2 Main Results

The semi-discrete model consists of a system of advection-reaction-diffusion equations on concentrations $c_k = c_k(t,y)$, velocities $u_k = u_k(t,y)$, pressures $p_k = p_k(t,y)$, describing motion of miscible liquids in several vertical tubes (n real lines, $y \in \mathbb{R}$, $k = 1,\dots,n$) and interflow between them (governed by velocities $u_{k+1/2}$).

Let $n \in \mathbb{N}, n \geq 2$, be the number of tubes. The *n-tubes IPM model* is obtained as a formal limit of the upwind finite-volume scheme and reads as follows, $k = 1,\dots,n$:



$$\text{(transport eq. in } k\text{-th tube)} \qquad \partial_t c_k + \partial_y(u_k c_k) - \nu \partial_{yy} c_k = f_{k-1/2} - f_{k+1/2}, \tag{1}$$

$$\text{(incompressibility condition)} \qquad l \cdot \partial_y u_k - u_{k-1/2} + u_{k+1/2} = 0. \tag{2}$$

Function $f_{k+1/2}$ is responsible for the interflow between k-th and $(k+1)$ tubes:

$$f_{k+1/2} = \begin{cases} c_k \cdot \frac{u_{k+1/2}}{l}, & u_{k+1/2} \geq 0, \text{ (fluid flows from tube } k \text{ to } (k+1)) \\ c_{k+1} \cdot \frac{u_{k+1/2}}{l}, & u_{k+1/2} \leq 0. \text{ (fluid flows from tube } (k+1) \text{ to } k) \end{cases} \tag{3}$$

The velocities u_k and $u_{k+1/2}$ are given by the Darcy's law:

$$u_k = -\partial_y p_k - c_k, \qquad\qquad u_{k+1/2} = \frac{p_{k+1} - p_k}{l}. \tag{4}$$

Here $l > 0$ is a parameter equal to the distance between the tubes. We assume that the last, n-th tube, is connected with the 1-st tube, thus all the indexes in the equations should be understood modulo n.

Numerical modelling shows that the typical asymptotic solution as $t \to \infty$ for initial data close to (2) for a small number of tubes looks like a stacked combination of traveling waves which we call a *propagating terrace*.

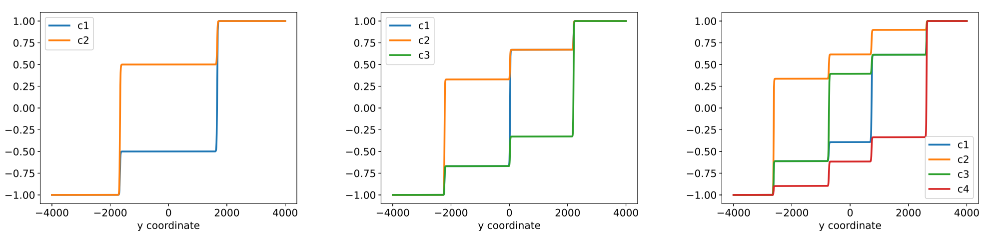


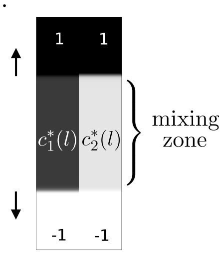
Figure 1: Typical asymptotic solution c_k, $k = 1,\dots,n$ for $n = 2,3,4$ tubes

In the presentation we give a rigorous justification of the existence of a propagating terrace in the simplest setting of two tubes — equations (1)-(4) for $n = 2$ (see [1]). The *main result* of [1] claims:

Theorem 2.1. *For fixed $\nu > 0$ and sufficiently small values of $l > 0$ there exist two intermediate concentrations $c_1^*(l) \in (-1,1)$, $c_2^*(l) \in (-1,1)$ and two traveling wave (TW) solutions of the system (1)–(4) that connect the states:*

$$(-1,-1) \xrightarrow{TW} (c_1^*(l), c_2^*(l)) \xrightarrow{TW} (1,1). \tag{5}$$

Moreover, the speeds of the traveling waves approach $-1/4$ and $1/4$ as $l \to 0$.



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