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# Geometric Measure Theory and Plateau's problem

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## Purpose of these lectures:

Informal description of some approaches to the solution of Plateau's problem and underlying tools

## Summary

1. Plateau's problem and the minimal surface equation
2. The parametric approach (Douglas, Radó)
3. Integral Currents (Federer + Fleming)
4. Set/measure theoretic approach (Reifenberg, Harrison + Pugh, revisited by De Lellis + Ghiraldin + Maggi)

# 1

# Plateau's problem and the minimal surface equation

# Plateau's problem

d-dimensional      d-dim. volume

Find the surface $\Sigma$ with minial area among those which spans a given boundary $\Gamma$ — d-1 dimens. in $\mathbb{R}^n$ (or another given ambient space).

## Remarks

- "find" = "prove the existence of"
- If surfaces are not smooth the meaning of "area" and "spans a given boundary" should be clarified

# The minimal surface equation

Let $\Sigma$ be a solution of Plateau's problem. Then it satisfies the Euler-Lagrange eq. associated to the area functional:

mean curvature (vectorfield) of $\Sigma$

$$H_\Sigma = 0$$

A surface $\Sigma$ with $H_\Sigma = 0$ is called a minimal surface.

## Idea of proof (everything is smooth)

Let $\eta$ be vectorfield on $\mathbb{R}^n$ null on $\partial\Sigma$

set $\Phi_t(x) := x + t\eta(x) \quad \forall x \in \mathbb{R}^n \ \forall t \in \mathbb{R}$

set $\Sigma_t := \Phi_t(\Sigma) \quad \forall t \in \mathbb{R}$; $\Sigma_t$ is a surface for small $t$ and $\partial\Sigma_t = \partial\Sigma$

We can compute the derivative of $t \mapsto \text{area}(\Sigma_t)$ at $t=0$:

d-dim. volume

first variation of the area of $\Sigma$ in direction $\eta$

$$\frac{d}{dt}\text{area}(\Sigma_t)\Big|_{t=0} = \int_\Sigma \eta(x)\cdot H_\Sigma(x)\, dx \qquad (*)$$

d-dim. volume measure on $\Sigma$

If $\Sigma$ solves Plateau's problem then

$t \mapsto \text{area}(\Sigma_t)$ has a **minimum** in $t=0$ and then

$$0 = \frac{d}{dt}\text{area}(\Sigma_t)\Big|_{t=0} = \int_\Sigma \eta\cdot H_\Sigma \quad \forall \eta \quad \Longrightarrow \quad H_\Sigma = 0$$

Proof of (*). Recall the area formula

$$\text{area}(\Sigma_t) = \int_\Sigma J_T \Phi_t \, dx$$

where $J_T \Phi_t$ is the (tangential) Jacobian of $\Phi_t$:

$$J_T \Phi_t(x) := \sqrt{\det\left[ \left(\nabla_T \Phi_t(x)\right)^T \left(\nabla_T \Phi_t(x)\right) \right]}$$

$$= 1 + t \, \text{div}_T \eta(x) + O(t^2)$$

Taylor expansion in $t$ (how is it done?)

Then

$$\text{area}(\Sigma_t) = \text{area}(\Sigma) + t \int_\Sigma \text{div}_T \eta + O(t^2)$$

Finally

$$\underbrace{\frac{d}{dt} \text{area}(\Sigma_t)\Big|_{t=0} = \int_\Sigma \text{div}_T \eta}_{\text{weak form of the first variation}} \overset{\text{divergence theorem on } \Sigma}{=} \int_\Sigma \eta \cdot H_\Sigma + \overset{0}{\overset{\shortparallel}{\cancel{\int_{\partial\Sigma} \eta \cdot \nu_{\partial\Sigma}}}}$$

$\square$

## 2

# Existence I :

## The parametric approach

# Existence proofs

Model problem: minimize the Dirichlet functional

$$E(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 dx$$

with boundary condition $u = u_0$ on $\partial\Omega$

STEP 1. Existence of minimizer in the Sobolev space $H^1_{u_0}(\Omega)$

STEP 2. Regularity of $u$ $\longrightarrow$ $u$ solves $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$

Proof of step 1. Take a minimizing seq. $(u_n)$ for $F(u)$ in $H^1_{u_0}$

(a) $(u_n)$ is bounded in $H^1_{u_0} \Longrightarrow u_n \xrightarrow{w} \bar{u} \in H^1_{u_0}$ up to subseq. [compactness]

(b) $\inf\{F(u) : u \in H^1_{u_0}\} = \lim_{n\to\infty} F(u_n) \geq F(\bar{u}) \Longrightarrow \bar{u}$ minimizes $F(u)$ on $H^1_{u_0}$

[semicontinuity]

# Passing to Plateau's problem...

the scheme is the same but there are problems with both steps (regardless of the approach).

Regularity fails in general. There exist $\Gamma$ s.t. every minimizing seq. $(\Sigma_n)$ converge to a limit surface $\overline{\Sigma}$ which is singular.

Example. In $\mathbb{R}^4 \simeq \mathbb{C}^2$ let $\Gamma$ be parametrized by $\gamma : \underset{\mathbb{C}}{\underset{\cap}{\mathbb{S}^1}} \longrightarrow \mathbb{C}^2$ where $\gamma(z) := (z^2, z^3)$. Then

$$\overline{\Sigma} := \{(z^2, z^3) : z \in \overset{\text{unit disk in } \mathbb{C}}{\mathbb{D}}\} \subset \overset{\text{singular at the origin}}{S} := \{w \in \mathbb{C}^2 : w_1^3 - w_2^2 = 0\}$$

Let $\Sigma$ be a "piece" of a complex surface (possibly singular). Then $\Sigma$ solves Plateau's problem with $\Gamma := \partial\Sigma$.

# The parametric approach

Let $\Sigma$ be a surface parametrized by $u : \underbrace{D}_{\text{reference domain in } \mathbb{R}^d} \longrightarrow \mathbb{R}^n$. Then

$$\text{area}(\Sigma) = \underbrace{\int_D Ju(x)\, dx}_{F(u)} \qquad Ju(x) = \sqrt{\det\left[(\nabla u)^T (\nabla u)\right]}$$

Let $u_0 : \partial D \longrightarrow \mathbb{R}^n$ parametrize $\Gamma$. Then $u = u_0$ on $\partial D \implies \partial\Sigma = \Gamma$.

Now $F$ is well-defined and weakly l.s.c. on $\underbrace{W^{1,p}}_{W^{1,p}(D;\mathbb{R}^n)}$ $\forall p > 1$ (GOOD NEWS!)

However, compactness does not hold for any $p > 1$, that is,

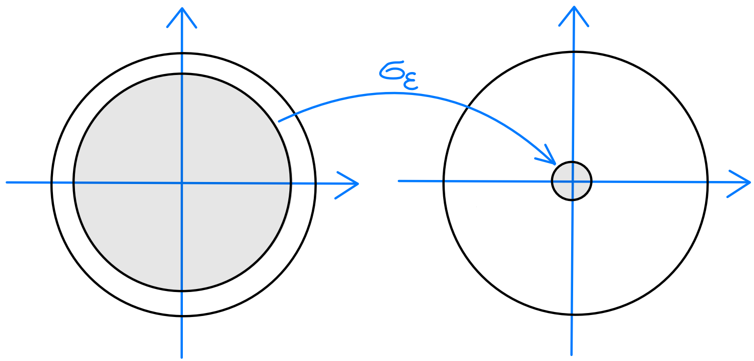
$(u_n)$ minimizing seq. for $F$ on $W^{1,p}_{u_0}$ $\not\Longrightarrow$ $(u_n)$ is bounded in $W^{1,p}$

(BAD NEWS!)

## Example

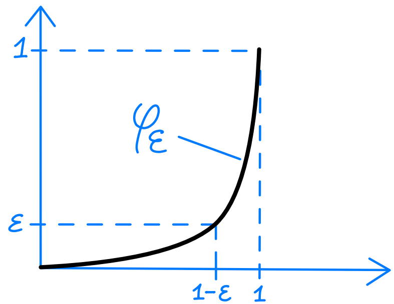
Fix $u : \overline{B} := \overline{B(0,1)} \to \mathbb{R}^n$ smooth parametrization of some $\Sigma$.

$\forall \varepsilon > 0$ take $\sigma_\varepsilon : \overline{B} \to \overline{B}$ diffeo s.t. $\sigma_\varepsilon(x) = x \ \forall x \in \partial B$, $\sigma_\varepsilon\big(B(0,1-\varepsilon)\big) = B(0,\varepsilon)$




for example $\sigma_\varepsilon(x) := \varphi_\varepsilon(|x|)\dfrac{x}{|x|}$

with




Set $u_\varepsilon := u \circ \sigma_\varepsilon$. Then $u_\varepsilon$ parametrizes $\Sigma$, then $F(u_\varepsilon) = \text{area}(\Sigma)$.

Moreover $u_\varepsilon \xrightarrow[\varepsilon \to 0]{} u(0)$ on $B$ but $u_{\varepsilon|\partial B} = u_{|\partial B} \not\to u(0)$.

constant map!

Something is wrong...

# Going around lack of compactness

Reminder : geodesics connecting two points on a Riemannian manifold are obtained by minimizing $E(\gamma) := \int_0^1 \frac{1}{2}|\dot\gamma|^2$ while length is $F(\gamma) := \int_0^1 |\dot\gamma|$ . How comes ?

A similar trick works in dimension $d=2$.

Let $E(u) := \int_B \frac{1}{2}|\nabla u|^2$ . Then

$$F(u) = \int_B |Ju| = \int_B \left|\frac{\partial u}{\partial x_1} \times \frac{\partial u}{\partial x_2}\right|$$

= holds iff $\frac{\partial u}{\partial x_1} \perp \frac{\partial u}{\partial x_2}$

$$\leqslant \int_B \left|\frac{\partial u}{\partial x_1}\right| \cdot \left|\frac{\partial u}{\partial x_2}\right| \leqslant \int_B \frac{1}{2}\left|\frac{\partial u}{\partial x_1}\right|^2 + \frac{1}{2}\left|\frac{\partial u}{\partial x_2}\right|^2 = E(u)$$

= holds iff $\left|\frac{\partial u}{\partial x_1}\right| = \left|\frac{\partial u}{\partial x_2}\right|$

**Lemma 1.** $F(u) \le E(u)$ and $=$ holds iff $\frac{\partial u}{\partial x_1} \perp \frac{\partial u}{\partial x_2}$ & $\left|\frac{\partial u}{\partial x_1}\right| = \left|\frac{\partial u}{\partial x_2}\right|$, that is, $\nabla u(x)$ is a conformal matrix $\forall x \in B$, that is, $u$ is a conformal map.

**Theorem 2 (Lichtenstein).** Given $u : \overline{B} \to \mathbb{R}^n$ there exists $\sigma : \overline{B} \to \overline{B}$ diffeo s.t. $\tilde{u} := u \circ \sigma$ is conformal. In particular $F(\tilde{u}) = E(\tilde{u})$.

**Corollary 3.** Let $\bar{u}$ minimize $E(u)$ with $u|_{\partial B}$ reparam. of $u_0$.
Then $\bar{u}$ minimizes $F(u)$ and is conformal.

**Proof.** $\forall u$:
$$F(u) \underset{\substack{\text{F invariant} \\ \text{under reparam.}}}{=} F(\tilde{u}) \underset{\text{Theor. 2}}{=} E(\tilde{u}) \ge E(\bar{u}) \underset{\text{Lemma 1}}{\ge} F(\bar{u}).$$

And for $u = \bar{u}$ we get $F(\bar{u}) \ge E(\bar{u}) \ge F(\bar{u}) \implies \bar{u}$ conformal. $\square$

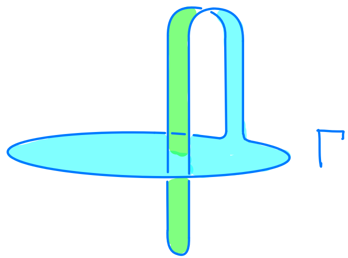
# Douglas – Radó approach

area functional    Dirichlet functional

You find a minimizer of $F(u)$ by minimizing $E(u)$ on $H^1(B,\mathbb{R}^n)$ with the constraint $u_{|\partial B}$ reparam. of $u_0$.
The minimizer $\bar{u}$ exists, is conformal and then harmonic.

## Remarks

- $\bar{u}$ is NOT injective $\longrightarrow$




- $\bar{u}$ is NOT an embedding. Ex.: $\bar{u}(z) := (z^2, z^3)$, $z \in B \subset \mathbb{C}$
- Accordingly the surface $\bar{\Sigma} := \bar{u}(\bar{B})$ is not regular.

- This approach works only if $d=2$ and the parametrization domain is the disk.

What is missing in the other cases is Lichtenstein theorem

In dimension $d>2$ conformal maps are scarce!

# 3

# Existence II : Integral Currents

## What do we need/want?

We start from the space $\mathcal{X}$ of regular d-dimensional surfaces in $\mathbb{R}^n$

The Key to existence is the construction of a suitable compactification $\overline{\mathcal{X}}$ of $\mathcal{X}$ that will play the role of the Sobolev space $H^1$ in the parametric approach.

"Suitable" means that the following properties hold:

$\boxed{1}$ "Compactness" Given a seq. $(\Sigma_n) \subset \overline{\mathcal{X}}$ such that

- $\partial \Sigma_n = \Gamma$ fixed
- $\text{area}(\Sigma_n) \leq C < +\infty$ — "area" = "d-dimen. volume"

Then $(\Sigma_n)$ converges up to subseq. to some $\Sigma \in \overline{\mathcal{X}}$ and

- $\partial \Sigma = \Gamma$

$\boxed{2}$ "Semicontinuity" If $\Sigma_n \longrightarrow \Sigma$ in $\overline{\mathcal{X}}$ then

- $\liminf\limits_{n\to\infty} \text{area}(\Sigma_n) \geqslant \text{area}(\Sigma)$

$\boxed{3}$ "Density of $\mathcal{X}$" $\forall\, \Sigma \in \overline{\mathcal{X}} \;\; \exists\, (\Sigma_n) \subset \mathcal{X}$ s.t.

- $\Sigma_n \longrightarrow \Sigma$ and $\text{area}(\Sigma_n) \longrightarrow \text{area}(\Sigma)$.

Watch out! We need to construct $\overline{\mathcal{X}}$ but also extend to $\overline{\mathcal{X}}$ the notions of area and boundary!!

Why $\boxed{3}$?

The space $I^d(\mathbb{R}^n)$ of d-dimensional integral currents in $\mathbb{R}^n$ satisfies all these requirements. A L M O S T

- $I^d(\mathbb{R}^n)$ is a compactification of oriented surfaces.
- It is not clear if regular oriented surfaces are dense. But at least polyhedral complexes are.

## Construction plan

- Hausdorff measure
- Rectiable sets
- Rectiable currents
- Boundary and integral currents
- Statements of main results

# Hausdorff measure

or $X$ metric space

even not integer

Given $d > 0$, the $d$-dimensional Hausdorff measure of $E \subset \mathbb{R}^n$ is

$$\mathcal{H}^d(E) := \sup_{\delta > 0} \mathcal{H}^d_\delta(E)$$

where

$$\mathcal{H}^d_\delta(E) := \frac{\alpha_d}{2^d} \sup \left\{ \sum_i (\operatorname{diam}(E_i))^d : \begin{array}{l} \{E_i\} \text{ countable cover of } E \\ \text{with } \operatorname{diam}(E_i) \leq \delta \ \forall n \end{array} \right\}$$

and $\alpha_d :=$ Volume of unit ball in $\mathbb{R}^d$.

## Remarks

- $\mathcal{H}^d$ is an outer meaure, Borel sets are $\mathcal{H}^d$-measurable
- The factor $\frac{\alpha_d}{2^d}$ implies that $\mathcal{H}^d =$ usual volume measure on every $d$-dimensional surface $\Sigma$ in $\mathbb{R}^n$.

- The set $E_i$ in the covering can be taken open/closed/convex and even balls if $E \subset$ d-dim. surface

- The Hausdorff dimension of $E = \dim_H(E)$ is the unique number s.t.

$$\mathcal{H}^d(E) = \begin{cases} +\infty & \text{if } d < \dim_H(E) \\ 0 & \text{if } d > \dim_H(E) \end{cases}$$

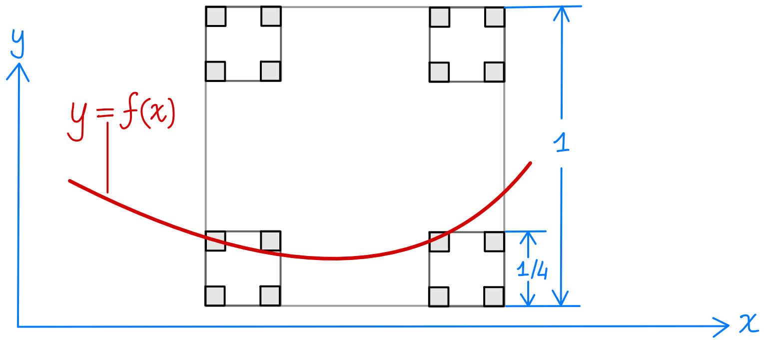
- Let $E$ be compact and self-similar, e.g., $E = \underbrace{\bigcup_{i=1}^{N}}_{\text{disjoint union!}} E_i$ with $E_i$ = copy of $E$ scaled by $\lambda$, rotated and translated. Then

$$d := \dim_H(E) = \frac{\log N}{\log(1/\lambda)}$$

Hint. $E = \dot{\bigcup_i} E_i \Rightarrow \mathcal{H}^d(E) = \sum_i \mathcal{H}^d(E_i) = N\lambda^d \mathcal{H}^d(E) \Rightarrow 1 = N\lambda^d \cdots$

More precisely $0 < \mathcal{H}^d(E) < +\infty$.

- Let $K_1$ be the standard Cantor set. Then $\dim_H(K_1) = \frac{\log 2}{\log 3}$.
- Let $K_2$ be the following set of Cantor type




Then $\dim_H(K_2) = 1$ and $0 < \mathcal{H}^1(K_2) < +\infty$.

However $\mathcal{H}^1(K_2 \cap \Gamma) = 0$ for every curve $\Gamma \subset \mathbb{R}^2$ of class $\mathcal{C}^1$

(Thus $K_2$ is an example of 1-purely unrectifiable set.)

Proof. Let $\Gamma$ = graph$(f)$ and let $p_x$ be the proj. on $x$ axis.
Then

$$\mathcal{H}^1(K_2 \cap \Gamma) = \int_{p_x(K_2 \cap \Gamma)} \sqrt{1+(f')^2}\, dx = 0 \qquad \square$$

$p_x(K_2 \cap \Gamma) \longleftarrow$ has measure 0

# Rectifiable sets

A set $E \subset \mathbb{R}^n$ with $\mathcal{H}^d(E) < +\infty$ is d-rectifiable if $E = \bigcup_{i=0}^{\infty} E_i$ where $\mathcal{H}^d(E_0) = 0$ and for $i > 0$, $E_i \subset S_i$ d-dimen. surface of class $\mathcal{C}^1$. E is 0-rectifiable if it is finite.

## Remarks

- Not the usual definition, but equivalent in $\mathbb{R}^n$.
- The set $K_2$ in prev. slide is $\mathcal{H}^1$ finite but not 1-rectifiable.
- Rectifiable sets can be quite "nasty", even dense in $\mathbb{R}^n$. Take for example $E := \bigcup_{i=1}^{\infty} S_i$ where $S_i$ is a d-dimen. disk with radius $r_i := 2^{-i}$ and center $x_i$ s.t. $\{x_i\}$ is dense in $\mathbb{R}^n$.

- Do we need such "awful" objects?

# Tangent planes

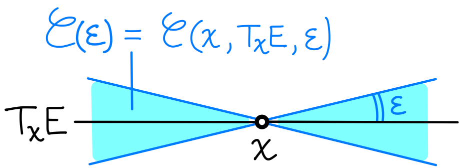
Let $E$ be $d$-rectifiable. Then for $\mathcal{H}^d$-a.e. $x \in E$ there exists an approximate $d$-dimensional tangent plane $T_xE$.

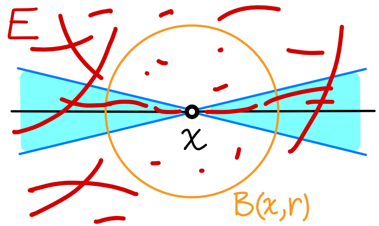
That is, $\forall\, \varepsilon > 0$

$$\mathcal{H}^d(E \cap B(x,r) \cap \mathscr{C}(\varepsilon)) \sim \alpha_d r^d; \quad \mathcal{H}^d(E \cap B(x,r) \cap (\mathscr{C}(\varepsilon))^c) \ll r^d \qquad \text{as } r \longrightarrow 0$$

where $\mathscr{C}(\varepsilon)$ is the cone

$$\mathscr{C}(\varepsilon) = \mathscr{C}(x, T_xE, \varepsilon) := x + \left\{ h \in \mathbb{R}^n : \operatorname{dist}(h, T_xE) \le |h| \sin \varepsilon \right\}$$




Moreover $T_xE = T_xS_i$ for $\mathcal{H}^d$-a.e. $x \in E \cap S_i$ and $i = 1, 2, \ldots$

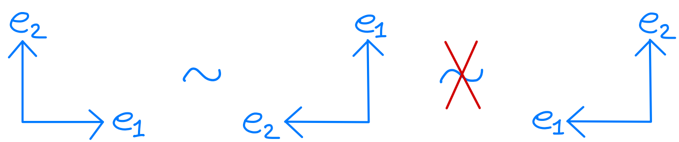
# Orientation of planes

Let $V$ be a $d$-plane in $\mathbb{R}^n$ and let $(e_1,\dots,e_d)$, $(e_1',\dots,e_d')$ be orthonormal bases of $V$.

We say that $(e_1,\dots,e_d)$ and $(e_1',\dots,e_d')$ induce the same orientation on $V$ if the change-of-basis matrix $M \in \mathbb{R}^{d\times d}$ satisfies $\det M > 0$. We write $(e_1,\dots,e_d) \sim (e_1',\dots,e_d')$

equivalence relation

Example:




Equivalence classes are (representeb by) simple $d$-vectors with norm 1, and denoted by $e_1 \wedge \cdots \wedge e_d$

# Covectors & differential forms

A d-covector on $\mathbb{R}^n$ is a function $\alpha : \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{d \text{ times}} \longrightarrow \mathbb{R}$ such that:

(i) $\alpha$ is linear in each variable;

(ii) $\alpha$ is alternating: $\forall i \neq j \ \forall v_1, \dots, v_d \in \mathbb{R}^d$ swapping $v_i$ and $v_j$ gives

$$\alpha(\dots, v_i, \dots, v_j, \dots) = -\alpha(\dots, v_j, \dots, v_i, \dots)$$

Important: from (i) and (ii) it follows that

$$(e_1, \dots, e_d) \sim (e_1', \dots, e_d') \implies \alpha(e_1, \dots, e_d) = \alpha(e_1', \dots, e_d')$$

Thus we simply write $\langle \alpha ; e_1 \wedge \cdots \wedge e_d \rangle$ for $\alpha(e_1, \dots, e_d)$.

A d-form on $\mathbb{R}^n$ is a map $\omega : x \in \mathbb{R}^n \longmapsto \omega(x)$ d-covector on $\mathbb{R}^n$

## Orientation of rectifiable sets (and surfaces)

Let $E$ be $d$-rectifiable. An orientation of $E$ is a map

$$\tau : x \in E \longmapsto e_1(x) \wedge \cdots \wedge e_d(x) \quad \text{orientation of } T_x E$$

- If $E$ is a regular surface, $\tau$ is usually required to be continuous
- We can integrate $d$-forms of oriented $d$-dimen. surfaces:

$$\int_S \omega = \int_S \langle \omega(x) ; \tau_S(x) \rangle \, d\mathcal{H}^d(x)$$

orientation of $S$

- It's the "right thing" to do because of Stokes' Theorem:

$$\int_{\partial S} \omega = \int_S d\omega$$

where $d\omega$ is the differential of $\omega$ $\left[ \omega = \sum_i \omega_i dx_i \Longrightarrow d\omega := \sum_i \sum_j \frac{\partial \omega_i}{\partial x_j} dx_j \wedge dx_i \right]$

# Rectifiable currents

A d-dimensional rectifiable current in $\mathbb{R}^n$ is a triple $T = [E, \tau, m]$ where

- $E$ is a d-rectifiable set,
- $\tau = e_1 \wedge \cdots \wedge e_d$ is an orientation of $E$,
- $m \in L^1(E, \mathcal{H}^d)$ is a multiplicity function.
- Why do we need multiplicity?

T has integral multiplicity if $m$ takes values in $\mathbb{Z}$.

T is polyhedral if $E$ is a finite union of d-dim. essentially disjoint simplexes $S_i$ and $\tau$ and $m$ are constant on each $S_i$.

## Integration of forms on currents

Given a bounded $d$-form $\omega$ on $\mathbb{R}^n$, the integral of $\omega$ on $T$ is

$$T(\omega) := \int_E \langle \omega(x) ; \tau(x) \rangle \, m(x) \, d\mathcal{H}^d(x)$$

In the background: definition of abstract currents as linear functionals on forms (of class $\mathcal{C}_c^\infty$).

## Convergence of currents

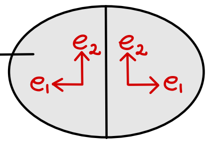
We say that $T_n \xrightarrow[n\to\infty]{} T$ if $T_n(\omega) \longrightarrow T(\omega)$ $\forall \omega$ of class $\mathcal{C}_c^\infty$

- Basically convergence in the sense of distributions
- There are other notions of convergence but ...

# Boundary

Given a d-dim. current $T$ and a $(d-1)$-dim. current $U$, we say that $U$ is the boundary of $T$, $U = \partial T$, if

$$U(\omega) = T(d\omega) \qquad \forall \omega \text{ of class } \mathcal{C}^\infty_c$$

- $S$ oriented regular surface $\Longrightarrow \partial[S, \tau_S, 1] = [\partial S, \tau_{\partial S}, 1]$ by Stokes' Thm.
- $T_n \longrightarrow T \Rightarrow \partial T_n \longrightarrow \partial T$ (stability of boundary)
- Question: what is $\partial[E, \tau, 1]$ if $E$ —  ?



- We need orientation to integrate forms, which we need to define the boundary $\partial T$, and convergence.

# Mass

The mass of T is

$$\mathbb{M}(T) := \int_E |m| \, d\mathcal{H}^d = \sup_{\|\omega\|_\infty \leq 1} T(\omega)$$

- $M(T) = \mathcal{H}^d(E)$ if $m = \pm 1$ a.e.
- $M(T)$ is lower semicontinuous in $T$.
- $M(T)$ can be defined also for abstract $T$.
- Mass is the desired extension of the area functional!!

# Integral currents

A rectifiable $d$-current $T = [E, \tau, m]$ is integral if:

- there exists a rectif. $(d-1)$-current $U = [E', \tau', m']$ s.t. $\partial T = U$;
- both $T$ and $U$ have integral multiplicity.

# Compactness theorem [Federer + Fleming]

Let $(T_n)$ be a sequence of $d$-dim. integral currents in $\mathbb{R}^n$ s.t.

(i) $\mathbb{M}(T_n) \leqslant C < +\infty$;

(ii) $\mathbb{M}(\partial T_n) \leqslant C < +\infty$.

Then $(T_n)$ converge up to subseq. to some integral current $T$.

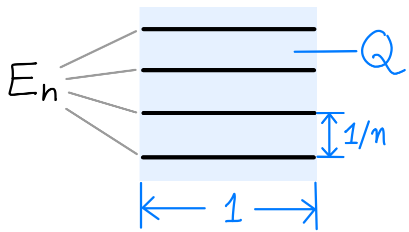
**Corollary.** Existence of an integral current $\bar{T}$ that minimizes $\mathbb{M}(T)$ under the constraint $\partial T = \Gamma$.

# Remarks

- Proofs of F&F Thm. are NOT based on results from Functional Analysis.
- Given a sequence $(T_n)$ of rectifiable currents with rectifiable boundaries s.t. (i), (ii) hold, then the limit of $T_n$ may be NOT rectifiable. (In F&F Thm. it is important that $T_n$ are integral.)

Consider indeed the following 1-currents in $\mathbb{R}^2$:

$T_n := [E_n, e, 1/n]$ with




and $e := \longrightarrow$

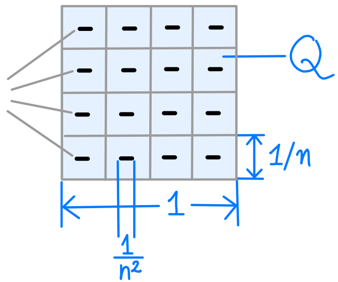
Then $\mathbb{M}(T_n) = 1$, $\mathbb{M}(\partial T_n) = 2$ $\forall n$, and

$$T_n(\omega) = \int_{E_n} \langle \omega(x) ; e \rangle \, \frac{1}{n} \, d\mathcal{H}^1(x) \xrightarrow[n \to \infty]{} \underbrace{\int_Q \langle \omega(x) ; e \rangle \, d\mathcal{L}^2(x)}_{\text{not a rectifiable current!}}$$

- Given a sequence $(T_n)$ of integral currents s.t. only (i) holds, then the limit of $T_n$ may be NOT rectifiable.

Consider indeed the following 1-currents in $\mathbb{R}^2$:

$T_n := [E_n, e, 1/n]$ with $E_n$




and $e := \longrightarrow$

Then $M(T_n) = 1$, $M(\partial T_n) = 2n^2$ $\forall n$, and

$$T_n(\omega) = \int_{E_n} \langle \omega(x) ; e \rangle \, d\mathcal{H}^1(x) \xrightarrow[n\to\infty]{} \underbrace{\int_Q \langle \omega(x) ; e \rangle \, d\mathcal{L}^2(x)}_{\text{same as before}}$$

not a rectifiable current!

# Some references


S.G. Krantz, H.R. Parks. Geometric Integration Theory. Birkhauser 2008.

Introduction to GMT focused on the theory of currents.

L. Simon. Lectures on Geometric Measure Theory. Australian National Univ. 1983

Introduction to GMT covering the theory of currents and varifolds.

Less detailed than Krantz & Parks

H. Federer. Geometric Measure Theory. Springer 1996 (reprint of 1st ed.)

Reference work on GMT and the theory of currents. Not a textbook, not for beginners.

F. Morgan. Geometric Measure Theory. A beginner's guide. Academic Press 2016.

A gentle introduction to Federer's book. Not fully detailed, but explains many ideas.


Other well known textbooks and reference works on GMT such as those authored by K. Falconer and P. Mattila may cover the theory of rectifiable sets (in some cases quite extensively) but not currents.